Small mass asymptotic for the motion with vanishing friction

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Abstract

We consider the small mass asymptotic (Smoluchowski-Kramers approximation) for the Langevin equation with a variable friction coefficient. The friction coefficient is assumed to be vanishing within certain region. We introduce a regularization for this problem and study the limiting motion for the 1-dimensional case and a multidimensional model problem. The limiting motion is a Markov process on a projected space. We specify the generator and boundary condition of this limiting Markov process and prove the convergence.

Keywords: Smoluchowski-Kramers approximation, diffusion processes, weak convergence, boundary theory of Markov processes.

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1 Introduction

The Langevin equation

$$\mu\ddot{\boldsymbol{q}}_t^\mu=\boldsymbol{b}(\boldsymbol{q}_t^\mu)-\lambda\dot{\boldsymbol{q}}_t^\mu+\sigma(\boldsymbol{q}_t^\mu)\dot{\boldsymbol{W}}_t\ ,\ \boldsymbol{q}_0^\mu=\boldsymbol{q}\in\mathbb{R}^n\ ,\dot{\boldsymbol{q}}_0^\mu=\boldsymbol{p}\in\mathbb{R}^n\ , \tag{1.1}$$

describes the motion of a particle of mass μ in a force field $\boldsymbol{b}(\boldsymbol{q})$, $\boldsymbol{q}\in\mathbb{R}^n$, subjected to random fluctuations and to a friction proportional to the velocity. Here $\boldsymbol{W}_t$ is the standard Wiener process in $\mathbb{R}^n$, $\lambda>0$ is the friction coefficient. The vector field $\boldsymbol{b}(\boldsymbol{q})$ and the matrix function $\sigma(\boldsymbol{q})$ are assumed to be continuously differentiable and bounded together with their first derivatives. The matrix $a(\boldsymbol{q})=(a_{ij}(\boldsymbol{q}))=\sigma(\boldsymbol{q})\sigma^*(\boldsymbol{q})$ is assumed to be non-degenerate.

It is assumed usually that the friction coefficient λ is a positive constant. Under this assumption, one can prove that $\boldsymbol{q}_t^\mu$ converges in probability as $\mu\downarrow 0$ uniformly on

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each finite time interval $[0,T]$ to an n-dimensional diffusion process $\boldsymbol{q}_t$: for any $\kappa, T > 0$ and any $\boldsymbol{p}_0^\mu = \boldsymbol{p} \in \mathbb{R}^n$, $\boldsymbol{q}_0^\mu = \boldsymbol{q} \in \mathbb{R}^n$ fixed,

$$\lim_{\mu \downarrow 0} \mathbf{P} \left(\max_{0 \le t \le T} |\boldsymbol{q}_t^\mu - \boldsymbol{q}_t|_{\mathbb{R}^d} > \kappa \right) = 0 \ .$$

Here $\boldsymbol{q}_t$ is the solution of equation

$$\dot{\boldsymbol{q}}_t = \frac{1}{\lambda}\boldsymbol{b}(\boldsymbol{q}_t) + \frac{1}{\lambda}\sigma(\boldsymbol{q}_t)\dot{\boldsymbol{W}}_t \ , \ \boldsymbol{q}_0 = \boldsymbol{q}_0^\mu = \boldsymbol{q} \in \mathbb{R}^n \ . \tag{1.2}$$

The stochastic term in (1.2) should be understood in the Itô sense.

The approximation of $\boldsymbol{q}_t^\mu$ by $\boldsymbol{q}_t$ for $0 < \mu << 1$ is called the Smoluchowski-Kramers approximation. This is the main justification for replacement of the second order equation (1.1) by the first order equation (1.2). The price for such a simplification, in particular, consists of certain non-universality of equation (1.2): The white noise in (1.1) is an idealization of a more regular stochastic process $\dot{\boldsymbol{W}}_t^\delta$ with correlation radius $\delta << 1$ converging to $\dot{\boldsymbol{W}}_t$ as $\delta \downarrow 0$. Let $\boldsymbol{q}_t^{\mu,\delta}$ be the solution of equation (1.1) with $\dot{\boldsymbol{W}}_t$ replaced by $\dot{\boldsymbol{W}}_t^\delta$. Then limit of $\boldsymbol{q}_t^{\mu,\delta}$ as $\mu, \delta \downarrow 0$ depends on the relation between μ and δ. Say, if first $\delta \downarrow 0$ and then $\mu \downarrow 0$, the stochastic integral in (1.2) should be understood in the Itô sense; if first $\mu \downarrow 0$ and then $\delta \downarrow 0$, $\boldsymbol{q}_t^{\mu,\delta}$ converges to the solution of (1.2) with stochastic integral in the Stratonovich sense. (See, for instance, [5].)

We considered in [6] the case of a variable friction coefficient $\lambda = \lambda(\boldsymbol{q})$. We assumed in that work that $\lambda(\boldsymbol{q})$ is smooth and $0 < \lambda_0 \le \lambda(\boldsymbol{q}) \le \Lambda < \infty$. It turns out that in this case the solution $\boldsymbol{q}_t^\mu$ of (1.1) does not converge, in general, to the solution of (1.2) with $\lambda = \lambda(\boldsymbol{q})$, so that the Smoluchowski-Kramers approximation should be modified. In order to do this, we considered in [6] equation (1.1) with $\dot{\boldsymbol{W}}_t$ replaced by $\dot{\boldsymbol{W}}_t^\delta$ described above:

$$\mu\ddot{\boldsymbol{q}}_t^{\mu,\delta} = \boldsymbol{b}(\boldsymbol{q}_t^{\mu,\delta}) - \lambda(\boldsymbol{q}_t^{\mu,\delta})\dot{\boldsymbol{q}}_t^{\mu,\delta} + \sigma(\boldsymbol{q}_t^{\mu,\delta})\dot{\boldsymbol{W}}_t^\delta \ , \ \boldsymbol{q}_0^{\mu,\delta} = \boldsymbol{q} \ , \ \dot{\boldsymbol{q}}_0^{\mu,\delta} = \boldsymbol{p} \ . \tag{1.3}$$

It was proved in [6] that after such a regularization, the solution of (1.3) has a limit $\boldsymbol{q}_t^\delta$ as $\mu \downarrow 0$, and $\boldsymbol{q}_t^\delta$ is the unique solution of the equation obtained from (1.3) as $\mu = 0$:

$$\dot{\boldsymbol{q}}_t^\delta = \frac{1}{\lambda(\boldsymbol{q}_t^\delta)}\boldsymbol{b}(\boldsymbol{q}_t^\delta) + \frac{1}{\lambda(\boldsymbol{q}_t^\delta)}\sigma(\boldsymbol{q}_t^\delta)\dot{\boldsymbol{W}}_t^\delta \ , \ \boldsymbol{q}_0^\delta = \boldsymbol{q} \ . \tag{1.4}$$

Now we can take $\delta \downarrow 0$ in (1.4). As the result we get the equation

$$\dot{\boldsymbol{q}}_t = \frac{1}{\lambda(\boldsymbol{q}_t)}\boldsymbol{b}(\boldsymbol{q}_t) + \frac{1}{\lambda(\boldsymbol{q}_t)}\sigma(\boldsymbol{q}_t) \circ \dot{\boldsymbol{W}}_t \ , \ \boldsymbol{q}_0 = \boldsymbol{q} \ , \tag{1.5}$$

where the stochastic term should be understood in the Stratonovich sense. We have, for any $\delta, \kappa, T > 0$ fixed and any $\boldsymbol{p}_0^{\mu,\delta} = \boldsymbol{p}$ fixed, that

$$\lim_{\mu \downarrow 0} \mathbf{P} \left(\max_{0 \le t \le T} |\boldsymbol{q}_t^{\mu,\delta} - \boldsymbol{q}_t^\delta|_{\mathbb{R}^d} > \kappa \right) = 0 \ ,$$

and we have

$$\lim_{\delta\to 0}\mathbf{E}\max_{t\in[0,T]}|\boldsymbol{q}_t^\delta-\boldsymbol{q}_t|_{\mathbb{R}^d}=0\ .$$

So the regularization leads to a modified Smoluchowski-Kramers equation (1.5).

In this paper we study a further generalization of the problem considered in [6]. Keeping the assumptions on uniform boundedness and smoothness of $\lambda(\bullet)$, we drop the assumption that $0<\lambda_0\leq\lambda(\boldsymbol{q})$ and instead assume that $\lambda(\boldsymbol{q})=0$ for $\boldsymbol{q}\in[G]\subset\mathbb{R}^n$ and $\lambda(\boldsymbol{q})>0$ for $\boldsymbol{q}\in\mathbb{R}^n\backslash[G]$. Here G is a domain in $\mathbb{R}^n$ and $[G]$ its closure in the standard Euclidean metric. For simplicity of presentation we assume in the rest of this paper that $\sigma(\bullet)$ is the identity matrix. (In Section 3 we further assume that $\boldsymbol{b}(\bullet)=\mathbf{0}$.) In order to use the results of [6] we introduce a further regularization of problem (1.5). We consider the problem

$$\dot{\boldsymbol{q}}_t^\varepsilon=\frac{1}{\lambda(\boldsymbol{q}_t^\varepsilon)+\varepsilon}\boldsymbol{b}(\boldsymbol{q}_t^\varepsilon)+\frac{1}{\lambda(\boldsymbol{q}_t^\varepsilon)+\varepsilon}\circ\dot{\boldsymbol{W}}_t\ ,\ \boldsymbol{q}_0^\varepsilon=\boldsymbol{q}\ ,\ \varepsilon>0 \tag{1.6}$$

and we study the limit of $\boldsymbol{q}_t^\varepsilon$ as $\varepsilon\downarrow 0$. This limiting process can be regarded as a limiting process of the system

$$\mu\ddot{\boldsymbol{q}}_t^{\mu,\delta,\varepsilon}=\boldsymbol{b}(\boldsymbol{q}_t^{\mu,\delta,\varepsilon})-(\lambda(\boldsymbol{q}_t^{\mu,\delta,\varepsilon})+\varepsilon)\dot{\boldsymbol{q}}_t^{\mu,\delta,\varepsilon}+\dot{\boldsymbol{W}}_t^\delta\ ,\ \boldsymbol{q}_0^{\mu,\delta,\varepsilon}=\boldsymbol{q}\ ,\ \dot{\boldsymbol{q}}_0^{\mu,\delta,\varepsilon}=\boldsymbol{p} \tag{1.7}$$

as first $\mu\downarrow 0$ then $\delta\downarrow 0$ and then $\varepsilon\downarrow 0$.

System (1.6), in Itô's form, can be written as follows:

$$\dot{\boldsymbol{q}}_t^\varepsilon=\frac{1}{\lambda(\boldsymbol{q}_t^\varepsilon)+\varepsilon}\boldsymbol{b}(\boldsymbol{q}_t^\varepsilon)-\frac{\nabla\lambda(\boldsymbol{q}_t^\varepsilon)}{2(\lambda(\boldsymbol{q}_t^\varepsilon)+\varepsilon)^3}+\frac{1}{\lambda(\boldsymbol{q}_t^\varepsilon)+\varepsilon}\dot{\boldsymbol{W}}_t\ ,\ \boldsymbol{q}_0^\varepsilon=\boldsymbol{q}\ . \tag{1.8}$$

However, as will be shown later, for non-compact region $[G]$, it is sometimes more convenient to consider the projection of the above system onto another space $\mathfrak{X}$. (In particular, in Section 3 the space $\mathfrak{X}$ is a cylinder $\mathfrak{X}=S^1\times[a-1,b+1]$ for $a<0,b>0$.) Let us work with system (1.8) on $\mathfrak{X}$ and compact region $[G]$. It turns out that, in the limit, to get a Markov process with continuous trajectories, one has to glue all the points of $[G]$ and form a projected space $\mathfrak{C}$. Let the projection map be $\boldsymbol{\pi}:\mathfrak{X}\to\mathfrak{C}$. We will prove, for the 1-dimensional case (Section 2) and a multidimensional model problem (Section 3), that the processes $\widetilde{\boldsymbol{q}}_t^\varepsilon=\boldsymbol{\pi}(\boldsymbol{q}_t^\varepsilon)$ converge weakly as $\varepsilon\downarrow 0$ to a continuous strong Markov process $\widetilde{\boldsymbol{q}}_t$ on $\mathfrak{C}$. We will characterize the generator of this Markov process and specify its boundary condition. In particular, we will show that as $\varepsilon>0$ is very small, certain mixing within $[G]$ is likely to happen for the process $\boldsymbol{q}_t^\varepsilon$. This mixing is the key mechanism that leads to our special boundary condition. We expect that (see Section 4), within the region that the friction is vanishing, similar mixing phenomenon will happen for the general multidimensional case.

It is worth mentioning here that some related problems are considered in [12], [13], [15] and [16]. It is also interesting to note that the limiting process for our two

dimensional model problem (see Section 3) shares some common feature with the so called Walsh's Brownian motion (see, for example [1]).

However, at this stage we are not able to prove, in the most general multidimensional case (except for the 2-d model problem in Section 3), the convergence of $\widetilde{\boldsymbol{q}}_t^\varepsilon = \boldsymbol{\pi}(\boldsymbol{q}_t^\varepsilon)$ in (1.8) to some Markov process $\widetilde{\boldsymbol{q}}_t$. We will formulate a conjecture about this in Section 4.

2 One dimensional case

Let us consider in this section the 1-dimensional case. Besides the usual assumptions made in Section 1 we suppose that our friction $\lambda(\bullet)$ satisfies $\lambda(q) > 0$ for $q \in (-\infty, -1) \cup (1, \infty)$. Let $\lambda(q) = 0$ for $q \in [-1, 1]$. Equation (1.8) now takes the following form:

$$\dot{q}_t^\varepsilon = \frac{b(q_t^\varepsilon)}{\lambda(q_t^\varepsilon)+\varepsilon} - \frac{\lambda'(q_t^\varepsilon)}{2(\lambda(q_t^\varepsilon)+\varepsilon)^3} + \frac{1}{\lambda(q_t^\varepsilon)+\varepsilon}\dot{W}_t \ , \ q_0^\varepsilon = q_0 \in \mathbb{R} \ . \tag{2.1}$$

We suppose that $q_0 \in [a-1, b+1]$ for some $a < 0 < b$. The process q_t^ε is supposed to be stopped once it hits $q = a-1$ or $q = b+1$.

Our goal is to study the asymptotic behavior of (2.1) as $\varepsilon \downarrow 0$. To this end we shall write the process (2.1) as a strong Markov process subject to a generalized second order differential operator in the form $D_{v^\varepsilon} D_{u^\varepsilon}$ (see [4], [2], [11]). We have

$$u^\varepsilon(q) = \int_0^q (\lambda(x)+\varepsilon) \exp\left(-2\int_0^x b(y)(\lambda(y)+\varepsilon)dy\right) dx \ , \tag{2.2}$$

$$v^\varepsilon(q) = 2\int_0^q (\lambda(x)+\varepsilon) \exp\left(2\int_0^x b(y)(\lambda(y)+\varepsilon)dy\right) dx \ . \tag{2.3}$$

For fixed $\varepsilon > 0$, the functions u^ε and v^ε are strictly increasing functions in their arguments. As $\varepsilon \downarrow 0$, they will converge uniformly on finite intervals to the functions u and v defined by

$$u(q) = \int_0^q \lambda(x) \exp\left(-2\int_0^x b(y)\lambda(y)dy\right) dx \ , \tag{2.4}$$

$$v(q) = 2\int_0^q \lambda(x) \exp\left(2\int_0^x b(y)\lambda(y)dy\right) dx \ . \tag{2.5}$$

The functions u and v are strictly increasing outside the interval $[-1, 1]$ and have constant stretches on $[-1, 1]$.

Consider a projection map π: we let $\pi([-1, 1]) = 0$ and $\pi(q) = q+1$ for $q < -1$ and $\pi(q) = q-1$ for $q > 1$. Consider the process $\widetilde{q}_t^\varepsilon = \pi(q_t^\varepsilon)$. Process $\widetilde{q}_t^\varepsilon$ for fixed $\varepsilon > 0$, in general, is *not* a Markov process.

Let us define two functions $\widetilde{u}$ and $\widetilde{v}$ as follows: $\widetilde{u}(\widetilde{q}) = u(\widetilde{q}-1)$ for $\widetilde{q} < 0$ and $\widetilde{u}(\widetilde{q}) = u(\widetilde{q}+1)$ for $\widetilde{q} > 0$ and $\widetilde{u}(0) = u(1) = u(-1) = 0$; $\widetilde{v}(\widetilde{q}) = v(\widetilde{q}-1)$ for $\widetilde{q} < 0$ and $\widetilde{v}(\widetilde{q}) = v(\widetilde{q}+1)$ for $\widetilde{q} > 0$ and $\widetilde{v}(0) = v(1) = v(-1) = 0$. Here the functions u and v are defined in (2.4), (2.5). The functions $\widetilde{u}$ and $\widetilde{v}$ are continuous strictly increasing functions on $[a,b]$.

Define a Markov process $\widetilde{q}_t$ on $[a,b]$ as follows. The generator A of $\widetilde{q}_t$ is $A = D_{\widetilde{v}} D_{\widetilde{u}}$. The domain of definition $D(A)$ of operator A consists of all functions f that are continuous on $[a,b]$, are twice continuously differentiable in $\widetilde{q} \in [a,b] \backslash \{0\}$, with finite limit $\lim\limits_{\widetilde{q}\to 0} Af(\widetilde{q})$ (taken as the value of $Af(0)$) and finite one-sided limits $\lim\limits_{\delta\downarrow 0} \dfrac{f(\delta)-f(0)}{\widetilde{u}(\delta)-\widetilde{u}(0)} \equiv D^+_{\widetilde{u}} f(0) = D^-_{\widetilde{u}} f(0) \equiv \lim\limits_{\delta\downarrow 0} \dfrac{f(0)-f(-\delta)}{\widetilde{u}(0)-\widetilde{u}(-\delta)}$. Also we have $\lim\limits_{\widetilde{q}\to a} Af(\widetilde{q}) = \lim\limits_{\widetilde{q}\to b} Af(\widetilde{q}) = 0$ (taken as the value of $Af(a)$ and $Af(b)$).

Lemma 2.1. *There exists the Markov process $\widetilde{q}_t$ on $[a,b]$.*

Proof. The existence of such a process could be checked similarly as in [7, Section 2]. For the sake of completeness and comparison with results in the next section we shall check it here. To this end we use an equivalent formulation of the Hille-Yosida theorem (see [7, Section 2] also [17, Theorem 2]). We check three conditions.

• The domain $D(A)$ is dense in the space $\mathbf{C}([a,b])$. This is because we can approximate every continuous function f with one that is constant in a neighborhood of 0. After that in the interior part of the intervals $[a,0)$ and $(0,b]$, at a positive distance from 0, with a smooth function. The approximating smooth function satisfy our boundary conditions since $Af(0) = D^+_{\widetilde{u}} f(0) = D^-_{\widetilde{u}} f(0) = 0$.

• The maximum principle: if $f \in D(A)$ and the function f reaches its maximum at a point $x_0 \in [a,b]$, then $Af(x_0) \le 0$. If $x_0 \ne 0$ we have $f'(x_0) = 0$ and $f''(x_0) \le 0$ and

$$D_{\widetilde{v}} D_{\widetilde{u}} f(x_0) = \frac{f''(x_0)}{\widetilde{v}'(x_0)\widetilde{u}'(x_0)} - \frac{\widetilde{u}''(x_0)}{\widetilde{v}'(x_0)(\widetilde{u}'(x_0))^2} f'(x_0) \le 0 \ .$$

If the maximum is achieved at 0, we consider the expansion

$$f(x) = f(0) + D_{\widetilde{u}} f(0)(\widetilde{u}(x) - \widetilde{u}(0)) + (Af(0) + o(1)) \int_0^x (\widetilde{v}(y) - \widetilde{v}(0)) d\widetilde{u}(y) \ .$$

The last integral is $O(\widetilde{u}(x)\widetilde{v}(x))$ as $x \to 0$. Since $D^-_{\widetilde{u}} f(0) \ge 0$ and $D^+_{\widetilde{u}} f(0) \le 0$, by our boundary conditions at 0 we get $D_{\widetilde{u}} f(0) = 0$. This implies that $Af(0) \le 0$.

• Existence of solution $f \in D(A)$ of $\lambda f - Af = F$ for all $F \in \mathbf{C}([a,b])$. On each of the intervals $[a,0)$ and $(0,b]$ the general solution of equation $\lambda f - D_{\widetilde{v}} D_{\widetilde{u}} f = F$, $F \in \mathbf{C}([a,b])$ can be written as

$$f^{\pm}(q) = \widehat{f}^{\pm}(q) + G^{\pm}(q) \ .$$

Here $\widehat{f}^{\pm}(q)$ satisfy the equation $\lambda \widehat{f}^{\pm} - D_{\widetilde{v}} D_{\widetilde{u}} \widehat{f}^{\pm} = F$, $\widehat{f}^{+}(0+) = 0$ (or $\widehat{f}^{-}(0-) = 0$), $D^{+}_{\widetilde{u}} \widehat{f}^{+}(0) = 0$ (or $D^{-}_{\widetilde{u}} \widehat{f}^{-}(0) = 0$) and $G^{\pm}(q)$ satisfy the equation $\lambda G^{\pm} - D_{\widetilde{v}} D_{\widetilde{u}} G^{\pm} = 0$, $G^{+}(0+) = k_1^{+}$ (or $G^{-}(0-) = k_1^{-}$), $D^{+}_{\widetilde{u}} G^{+}(0) = k_2^{+}$ (or $D^{-}_{\widetilde{u}} G^{-}(0) = k_2^{-}$). Here $k_1^{\pm}$ and $k_2^{\pm}$ are constants. Our boundary condition gives $k_1^{+} = k_1^{-}$ and $k_2^{+} = k_2^{-}$. The boundary condition $D_{\widetilde{u}} D_{\widetilde{v}} f^{+}(a) = D_{\widetilde{u}} D_{\widetilde{v}} f^{-}(b) = 0$ singles out a unique $f \in D(A)$. $\square$

We have

Theorem 2.1. *As $\varepsilon \downarrow 0$, for fixed $T > 0$, the process $\widetilde{q}^{\varepsilon}_t$ converges weakly in the space $\mathbf{C}_{[0,T]}([a,b])$ to the process $\widetilde{q}_t$.*

The proof of this Theorem is based on an application of the machinery developed in [8, Ch.8], [9] and [7]. We shall use the following lemma, which is the Lemma 3.1 of [8, Ch.8, page 301]. We formulate it here in the terminology that meets our purpose.

Lemma 2.2. *Let M be a metric space; Y, a continuous mapping $M \mapsto Y(M)$, $Y(M)$ being a complete separable metric space. Let $(X^{\varepsilon}_t, \mathbf{P}^{\varepsilon}_x)$ be a family of Markov processes in M; suppose that the process $Y(X^{\varepsilon}_t)$ has continuous trajectories. Let $(y_t, \mathbf{P}_y)$ be a Markov process with continuous paths in $Y(M)$ whose infinitesimal operator is A with domain of definition $D(A)$. Let $T > 0$. Let us suppose that the space $\mathbf{C}_{[0,T]}(Y(M))$ of continuous functions on $[0,T]$ with values in $Y(M)$ is taken as the sample space, so that the distribution of the process in the space of continuous functions is simply $\mathbf{P}_y$. Let Ψ be a subset of the space $\mathbf{C}_{[0,\infty)}(Y(M))$ such that for measures μ_1, μ_2 on $Y(M)$ the equality $\displaystyle\int F d\mu_1 = \int F d\mu_2$ for all $F \in \Psi$ implies $\mu_1 = \mu_2$. Let D be the subset of $D(A)$ such that for every $F \in \Psi$ and $\lambda > 0$ the equation $\lambda f - Af = F$ has a solution $f \in D$.*

Suppose that for every $x \in M$ the family of distributions $\mathbf{Q}^{\varepsilon}_x$ of $Y(X^{\varepsilon}_{\bullet})$ in the space $\mathbf{C}_{[0,T]}(Y(M))$ corresponding to the probabilities of $\mathbf{P}^{\varepsilon}_x$ is weakly pre-compact; and that for every compact $K \subset Y(M)$, for every $f \in D$ and every $\lambda > 0$,

$$\mathbf{E}^{\varepsilon}_x \int_0^{\infty} e^{-\lambda t} [\lambda f(Y(X^{\varepsilon}_t)) - Af(Y(X^{\varepsilon}_t))] dt \to f(Y(x))$$

as $\varepsilon \downarrow 0$ uniformly in $x \in Y^{-1}(K)$.

Then $\mathbf{Q}^{\varepsilon}_x$ converges weakly as $\varepsilon \downarrow 0$ to the probability measure $\mathbf{P}_{Y(x)}$.

Proof of Theorem 2.1. Making use of Lemma 2.2, we take the metric space $M = [a-1, b+1]$ and the mapping $Y = \pi$. The space $Y(M) = \pi([a-1, b+1]) = [a, b]$. We take the process q_t^ε as $(X_t^\varepsilon, \mathbf{P}_x^\varepsilon)$. We take the process $\widetilde{q}_t$ as $(y_t, \mathbf{P}_y)$.

Let Ψ be the space of all continuous bounded functions in $[a, b]$ which are once continuously differentiable inside $[a, 0)$ and $(0, b]$, with bounded derivatives. The space $D \subset D(A)$ consists of those functions $f \in D(A)$ such that they are continuous and bounded in $[a, b]$ and are three times continuously differentiable inside $[a, 0)$ and $(0, b]$, with bounded derivatives up to the third order.

Pre-compactness of the family of distributions of the process $\{\widetilde{q}_\bullet^\varepsilon\}_{\varepsilon>0}$ is checked in Lemma 2.4. What remains to do is to check that for every compact $K \subset [a, b]$, for every $f \in D$ and every $\lambda > 0$,

$$\mathbf{E}_{q_0}\left[\int_0^\infty e^{-\lambda t}[\lambda f(\pi(q_t^\varepsilon)) - Af(\pi(q_t^\varepsilon))]dt - f(\pi(q_0))\right] \to 0$$

as $\varepsilon \downarrow 0$ uniformly in $q_0 \in \pi^{-1}(K)$. This is done in Lemma 2.5. This finishes the proof of Theorem 2.1. $\square$

For positive δ small enough, let $G(\delta) = [a-1, -1-\delta] \cup [1+\delta, b+1]$. Let $0 < \delta' < \delta$. Let $C(\delta') = \{-1-\delta', 1+\delta'\}$. We introduce a sequence of stopping times $\tau_0 \leq \sigma_0 < \tau_1 < \sigma_1 < \tau_2 < \sigma_2 < ...$ by

$$\tau_0 = 0 \ , \ \sigma_n = \min\{t \geq \tau_n, q_t^\varepsilon \in G(\delta)\} \ , \ \tau_n = \min\{t > \sigma_{n-1} : q_t^\varepsilon \in C(\delta')\} \ .$$

This is well-defined up to some σ_k $(k \geq 0)$ such that

$$\mathbf{P}_{q_{\sigma_k}^\varepsilon}(q_{t+\sigma_k}^\varepsilon \text{ hits } a-1 \text{ or } b+1 \text{ before it hits } -1-\delta' \text{ or } 1+\delta') = 1 \ .$$

We will then define $\tau_{k+1} = \min\{t > \sigma_k : q_t^\varepsilon = a-1 \text{ or } b+1\}$. And we define $\tau_{k+1} < \sigma_{k+1} = \tau_{k+1} + 1 < \tau_{k+2} = \tau_{k+1} + 2 < \sigma_{k+2} = \tau_{k+1} + 3 < ...$ and so on.

We have $\lim\limits_{n\to\infty} \tau_n = \lim\limits_{n\to\infty} \sigma_n = \infty$. And we have obvious relations $q_{\tau_n}^\varepsilon \in C(\delta')$, $q_{\sigma_n}^\varepsilon \in C(\delta)$ for $1 \leq n \leq k$ (as long as $k \geq 1$, if $k = 0$ the process may start from $G(\delta)$ and goes directly to $a-1$ or $b+1$ without touching $C(\delta')$ and is stopped there, or it may start from $(-1-\delta, 1+\delta)$, reaches $\{-1-\delta, 1+\delta\}$ first and then goes directly to $a-1$ or $b+1$ without touching $C(\delta')$ and is stopped there). Also, for $n \geq k+1$ we have $q_{\tau_n}^\varepsilon = q_{\sigma_n}^\varepsilon = a-1$ or $b+1$. If $q_0^\varepsilon = q_0 \in G(\delta)$, then we have $\sigma_0 = 0$ and τ_1 is the first time at which the process q_t^ε reaches $C(\delta')$ or $\{a-1, b+1\}$.

Now we check weak pre-compactness of the family of distributions of the processes $\{\widetilde{q}_t^\varepsilon\}_{\varepsilon>0}$. To this end we need the following lemma, which is Lemma 5.1 in [7]. We formulate it using our terminology.

Lemma 2.3. *Let $\widetilde{q}_{\bullet}^{\varepsilon,\delta}$ for every $\varepsilon > 0$, $\delta > 0$, be a random element in $\mathbf{C}_{[0,T]}([a,b])$ such that $\max\limits_{0\le t\le T} |\widetilde{q}_t^{\varepsilon} - \widetilde{q}_t^{\varepsilon,\delta}| \le \delta$ on the whole probability space. If for every positive δ the family of distributions of $\widetilde{q}_{\bullet}^{\varepsilon,\delta}$, $\varepsilon > 0$, is tight, then the family of distributions of $\widetilde{q}_{\bullet}^{\varepsilon}$ is pre-compact.*

Now we have

Lemma 2.4. *The family of distributions of $\{\widetilde{q}_{\bullet}^{\varepsilon}\}_{\varepsilon>0}$ is pre-compact.*

Proof. Let $\delta' = \delta/2$ so that we need only one parameter δ. Between the times σ_{i-1} and τ_i the process q_t^ε is either in $[a, -1-\delta/2)$ or in $(1+\delta/2, b]$, and for $\sigma_{i-1} \le t < t' < \tau_i$ we have $|\widetilde{q}_t^{\varepsilon} - \widetilde{q}_{t'}^{\varepsilon}| = |q_t^\varepsilon - q_{t'}^\varepsilon|$. Since we have

$$q_t^\varepsilon - q_{t'}^\varepsilon = \int_t^{t'} \left[\frac{b(q_s^\varepsilon)}{\lambda(q_s^\varepsilon)+\varepsilon} - \frac{\lambda'(q_s^\varepsilon)}{2(\lambda(q_t^\varepsilon)+\varepsilon)^3}\right] ds + \int_t^{t'} \frac{1}{\lambda(q_s^\varepsilon)+\varepsilon} dW_s \ ,$$

we can estimate

$$\mathbf{E}|q_t^\varepsilon - q_{t'}^\varepsilon|^4 \le K(\delta)|t-t'|^2 \ .$$

The constant $K(\delta)$ is independent of ε provided that ε is small. Now we let

$$Z_t^{\varepsilon,\delta} = \int_0^t \mathbf{1}_{G(\delta/2)}(q_s^\varepsilon) \left[\frac{b(q_s^\varepsilon)}{\lambda(q_s^\varepsilon)+\varepsilon} - \frac{\lambda'(q_s^\varepsilon)}{2(\lambda(q_t^\varepsilon)+\varepsilon)^3}\right] ds + \int_0^t \mathbf{1}_{G(\delta/2)}(q_s^\varepsilon) \frac{1}{\lambda(q_s^\varepsilon)+\varepsilon} dW_s \ .$$

From the above estimate we see that $Z_t^{\varepsilon,\delta}$ for fixed $\delta > 0$ is tight. The trajectories of these stochastic processes satisfy the Hölder condition $|Z_t^{\varepsilon,\delta} - Z_{t'}^{\varepsilon,\delta}| \le H^{\varepsilon,\delta}|t-t'|^{1/5}$ where $H^{\varepsilon,\delta}$ are random variables with $\mathbf{E}(H^{\varepsilon,\delta})^4$ bounded by the same $K(\delta)$.

For $i \ge 1$ if $q_{\tau_i}^\varepsilon \in C(\delta/2)$ and $q_{\sigma_i}^\varepsilon \in C(\delta)$ then between the times τ_i and σ_i $(\le T)$ the process q_t^ε travels a distance at least $\delta/2$ and at least this distance in $G(\delta/2)$ on the same interval either $[a, -1-\delta/2)$ or $(1+\delta/2, b]$. By our estimate on Hölder continuity of $Z_t^{\varepsilon,\delta}$ this implies that $\sigma_i - \tau_i \ge \left(\dfrac{\delta}{4H^{\varepsilon,\delta}}\right)^5$, $i \ge 1$. If $q_{\tau_i}^\varepsilon \in \{a-1, b+1\}$ then by our definition of the stopping time $\sigma_i = \tau_i + 1$ we can choose δ small enough such that the above inequality also holds.

Now we shall define the process $\widetilde{q}_t^{\varepsilon,\delta}$ as follows.

- For $\sigma_{i-1} \le t \le \tau_i$ we take $\widetilde{q}_t^{\varepsilon,\delta} = \widetilde{q}_t^{\varepsilon}$.
- For $\tau_0 \le t \le \sigma_0$ we take $\widetilde{q}_t^{\varepsilon,\delta} = \widetilde{q}_{\sigma_0}^{\varepsilon}$. This gives $\max\limits_{\tau_0\le t\le\sigma_0} |\widetilde{q}_t^{\varepsilon,\delta} - \widetilde{q}_t^{\varepsilon}| = \max\limits_{\tau_0\le t\le\sigma_0} |\widetilde{q}_{\sigma_0}^{\varepsilon} - \widetilde{q}_t^{\varepsilon}| \le \delta$.
- If $\tau_i < T < \sigma_i$ we take $\widetilde{q}_t^{\varepsilon,\delta} = \widetilde{q}_{\tau_i}^{\varepsilon}$ for $\tau_i \le t \le T$. This gives $\max\limits_{\tau_i\le t\le T} |\widetilde{q}_t^{\varepsilon,\delta} - \widetilde{q}_t^{\varepsilon}| = \max\limits_{\tau_i\le t\le T} |\widetilde{q}_{\tau_i}^{\varepsilon} - \widetilde{q}_t^{\varepsilon}| \le \delta/2$.

• If $\sigma_i \le T$. In this case if $\widetilde{q}^{\varepsilon}_{\tau_i}$ and $\widetilde{q}^{\varepsilon}_{\sigma_i}$ are within a distance $\le \delta$ from 0, we define $\widetilde{q}^{\varepsilon,\delta}_{\frac{\tau_i+\sigma_i}{2}} = 0$,

$$\widetilde{q}^{\varepsilon,\delta}_t = \left(1 - \frac{2(t-\tau_i)}{\sigma_i - \tau_i}\right) \widetilde{q}^{\varepsilon}_{\tau_i} \text{ for } \tau_i \le t \le \frac{\tau_i + \sigma_i}{2} \ ,$$

$$\widetilde{q}^{\varepsilon,\delta}_t = -\left(1 - \frac{2(t-\tau_i)}{\sigma_i - \tau_i}\right) \widetilde{q}^{\varepsilon}_{\sigma_i} \text{ for } \frac{\tau_i + \sigma_i}{2} \le t \le \sigma_i \ .$$

Since this is just a linear interpolation it is clear that in this case we have $\max\limits_{\tau_i \le t \le \sigma_i} |\widetilde{q}^{\varepsilon,\delta}_t - \widetilde{q}^{\varepsilon}_t| \le 2\delta$. Within this time interval $\tau_i \le t < t' \le \sigma_i$, $i \ge 1$ we have

$$|\widetilde{q}^{\varepsilon,\delta}_t - \widetilde{q}^{\varepsilon,\delta}_{t'}| \le \frac{\delta}{|\sigma_i - \tau_i|}|t-t'| \le \frac{\delta}{(\min\limits_{i \ge 1} \frac{1}{2}|\sigma_i - \tau_i|)^{1/5}}|t-t'|^{1/5} \le 2^{11/5} H^{\varepsilon,\delta}|t-t'|^{1/5} \ .$$

Another possibility is that $q^{\varepsilon}_{\sigma_i} = q^{\varepsilon}_{\tau_i} = a-1$ or $b+1$. In this case we define $\widetilde{q}^{\varepsilon,\delta}_t = \widetilde{q}^{\varepsilon}_t$ for $\tau_i \le t < \sigma_i$.

On the whole interval $0 \le t < t' \le T$ we have $|\widetilde{q}^{\varepsilon,\delta}_t - \widetilde{q}^{\varepsilon,\delta}_{t'}| \le (2^{11/5}+2)H^{\varepsilon,\delta}|t'-t|^{1/5}$ for $|t'-t| \le \left(\dfrac{\delta}{4H^{\varepsilon,\delta}}\right)^5$. This means that for fixed $\delta > 0$ we have the tightness of the family of distributions of $\widetilde{q}^{\varepsilon,\delta}_t$ in the space $\mathbf{C}_{[0,T]}([a,b])$. Since we checked $\max\limits_{0 \le t \le T} |\widetilde{q}^{\varepsilon,\delta}_t - \widetilde{q}^{\varepsilon}_t| \le 2\delta$, by using Lemma 2.3 with 2δ instead of δ we get the pre-compactness of the family of distributions of $\widetilde{q}^{\varepsilon}_t$ in $\mathbf{C}_{[0,T]}([a,b])$. $\square$

The proof of the next Lemma 2.5 is based on Lemmas 2.6-2.10. Within the proof of this lemma and the auxiliary Lemmas 2.6-2.10, we will take $\varepsilon \downarrow 0$, $\delta = \delta(\varepsilon) \downarrow 0$, $\delta' = \delta'(\varepsilon) \downarrow 0$ in an asymptotic order such that $0 < \varepsilon << \delta' << \delta$. Although not very precise, but for simplicity of presentation we will just refer this choice of order as first $\varepsilon \downarrow 0$, then $\delta' \downarrow 0$ and then $\delta \downarrow 0$. It could be checked that such an order of taking limit does not alter the validity of the result.

Throughout the rest of this section and next section when we use symbols U, V, M_i, C_i, A_i, etc., they are referring to some positive constants. We will not point out this explicitly unless some special properties of the implied constants are stressed. Also we sometimes use the same letter for constants in different estimates.

Lemma 2.5. *For every compact $K \subset [a,b]$, for every $f \in D$ and every $\lambda > 0$,*

$$\mathbf{E}_{q_0}\left[\int_0^\infty e^{-\lambda t}[\lambda f(\pi(q^{\varepsilon}_t)) - Af(\pi(q^{\varepsilon}_t))]dt - f(\pi(q_0))\right] \to 0$$

as $\varepsilon \downarrow 0$ uniformly in $q_0 \in \pi^{-1}(K)$.

Proof. The above expectation can be written as

$$
\begin{aligned}
&\mathbf{E}_{q_0}\left[\sum_{n=0}^{\infty}\left[\int_{\tau_n}^{\sigma_n} e^{-\lambda t}[\lambda f(\pi(q_t^\varepsilon)) - Af(\pi(q_t^\varepsilon))]dt + e^{-\lambda\sigma_n} f(\pi(q_{\sigma_n}^\varepsilon)) - e^{-\lambda\tau_n} f(\pi(q_{\tau_n}^\varepsilon))\right] + \right.\\
&\left.\sum_{n=0}^{\infty}\left[\int_{\sigma_n}^{\tau_{n+1}} e^{-\lambda t}[\lambda f(\pi(q_t^\varepsilon)) - Af(\pi(q_t^\varepsilon))]dt + e^{-\lambda\tau_{n+1}} f(\pi(q_{\tau_{n+1}}^\varepsilon)) - e^{-\lambda\sigma_n} f(\pi(q_{\sigma_n}^\varepsilon))\right]\right]\\
&= \mathbf{E}_{q_0}\left[\sum_{n=0}^{\infty} e^{-\lambda\tau_n}\psi_1^\varepsilon(q_{\tau_n}^\varepsilon) + \sum_{n=0}^{\infty} e^{-\lambda\sigma_n}\psi_2^\varepsilon(q_{\sigma_n}^\varepsilon)\right] ,
\end{aligned} \tag{2.6}
$$

where

$$
\psi_1^\varepsilon(q) = \mathbf{E}_q\left[\int_0^{\sigma_0} e^{-\lambda t}[\lambda f(\pi(q_t^\varepsilon)) - Af(\pi(q_t^\varepsilon))]dt + e^{-\lambda\sigma_0} f(\pi(q_{\sigma_0}^\varepsilon))\right] - f(\pi(q)) , \tag{2.7}
$$

$$
\psi_2^\varepsilon(q) = \mathbf{E}_q\left[\int_{\sigma_0}^{\tau_1} e^{-\lambda t}[\lambda f(\pi(q_t^\varepsilon)) - Af(\pi(q_t^\varepsilon))]dt + e^{-\lambda\tau_1} f(\pi(q_{\tau_1}^\varepsilon))\right] - f(\pi(q)) . \tag{2.8}
$$

We used the strong Markov property of q_t^ε. Since for $n \geq k+1$ we have $\psi_1^\varepsilon(q_{\tau_n}^\varepsilon) = \psi_2^\varepsilon(q_{\sigma_n}^\varepsilon) = 0$ we can assume that the function ψ_2^ε is taken at a point on $G(\delta)\backslash\{a-1, b+1\}$ and the expectation is determined by the values of the process q_t^ε in one of the intervals either $(1+\delta', b+1]$ or $[a-1, -1-\delta')$. We will prove, in Lemma 2.6, that under our specified asymptotic order we can have $|\psi_2^\varepsilon(q)| \leq (\widetilde{u}(\delta) - \widetilde{u}(-\delta))^2$ as $\varepsilon \downarrow 0$.

We can assume that the function ψ_1^ε is taken at a point in $[-1-\delta', 1+\delta']$ (in the case when $n = 0$ and $q_0^\varepsilon \in G(\delta)$, we also have $\psi_1^\varepsilon(q_0) = 0$). We can write

$$
\begin{aligned}
&\psi_1^\varepsilon(q)\\
&= \left(\mathbf{E}_q f(\pi(q_{\sigma_0}^\varepsilon)) - f(\pi(q))\right) - \mathbf{E}_q(1 - e^{-\lambda\sigma_0}) f(\pi(q_{\sigma_0}^\varepsilon)) + \mathbf{E}_q \int_0^{\sigma_0} e^{-\lambda t}[\lambda f(\pi(q_t^\varepsilon)) - Af(\pi(q_t^\varepsilon)))]dt\\
&= (I)^\varepsilon(q) + (II)^\varepsilon(q) + (III)^\varepsilon(q) .
\end{aligned} \tag{2.9}
$$

We are going to prove, in Lemma 2.8, that for $q \in [-1-\delta', 1+\delta']$, for a function $f \in D$ we can have the estimate $|(I)^\varepsilon(q)| \leq M_1(\widetilde{u}(\delta) - \widetilde{u}(-\delta))^2$.

In Lemma 2.9 we will show that $\mathbf{E}_q\sigma_0 \leq M_1(\widetilde{u}(\delta) - \widetilde{u}(-\delta))(\widetilde{v}(\delta) - \widetilde{v}(-\delta))$ and $\mathbf{E}_q(1 - e^{-\lambda\sigma_0}) \leq M_1(\widetilde{u}(\delta) - \widetilde{u}(-\delta))(\widetilde{v}(\delta) - \widetilde{v}(-\delta))$ so that $|(II)^\varepsilon(q)| + |(III)^\varepsilon(q)| < M_1(\widetilde{u}(\delta) - \widetilde{u}(-\delta))(\widetilde{v}(\delta) - \widetilde{v}(-\delta))$ for $q \in [-1-\delta', 1+\delta']$.

These estimates show that

$$
|\psi_1^\varepsilon(q)| < (\widetilde{u}(\delta) - \widetilde{u}(-\delta))^2 + M_1(\widetilde{u}(\delta) - \widetilde{u}(-\delta))(\widetilde{v}(\delta) - \widetilde{v}(-\delta))
$$

for all $q \in [-1-\delta', 1+\delta']$.

As we only consider the arguments $q^\varepsilon_{\tau_n}$ of ψ^ε_1 in (2.6) being in $[-1-\delta', 1+\delta']$ starting with $n=1$ (otherwise $\psi^\varepsilon_1 = 0$), we have, by strong Markov property of q^ε_t, that

$$\begin{aligned}
&\left|\mathbf{E}_{q_0}\sum_{n=1}^\infty e^{-\lambda\tau_n}\psi^\varepsilon_1(q^\varepsilon_{\tau_n})\right| \\
&\leq (\widetilde{u}(\delta)-\widetilde{u}(-\delta))^2 + M_1(\widetilde{u}(\delta)-\widetilde{u}(-\delta))(\widetilde{v}(\delta)-\widetilde{v}(-\delta))\sum_{n=1}^\infty \mathbf{E}_{q_0}e^{-\lambda\tau_n} \\
&\leq (\widetilde{u}(\delta)-\widetilde{u}(-\delta))^2 + M_1(\widetilde{u}(\delta)-\widetilde{u}(-\delta))(\widetilde{v}(\delta)-\widetilde{v}(-\delta))\sum_{n=1}^\infty \left(\sup_{q\in G(\delta)} \mathbf{E}_q e^{-\lambda\tau_1}\right)^{n-1} .
\end{aligned}$$

We will show, in Lemma 2.10, that $\mathbf{E}_q e^{-\lambda\tau_1} < 1 - M_2\widetilde{u}(\delta)\wedge(-\widetilde{u}(-\delta))$ for all $q \in G(\delta)$. Since as $\delta \downarrow 0$ we have $0 < M_2 \leq \left|\dfrac{\widetilde{u}(\delta)}{-\widetilde{u}(-\delta)}\right| \leq M_3 < \infty$, we have

$$\begin{aligned}
&\left|\mathbf{E}_{q_0}\sum_{n=1}^\infty e^{-\lambda\tau_n}\psi^\varepsilon_1(q^\varepsilon_{\tau_n})\right| \\
&\leq ((\widetilde{u}(\delta)-\widetilde{u}(-\delta))^2 + M_1(\widetilde{u}(\delta)-\widetilde{u}(-\delta))(\widetilde{v}(\delta)-\widetilde{v}(-\delta)))\frac{1}{M_2(\widetilde{u}(\delta))\wedge(-\widetilde{u}(-\delta))} \to 0
\end{aligned}$$

as $\delta \downarrow 0$. For $n=0$ the expectation $\mathbf{E}_{q_0}\psi^\varepsilon_1(q^\varepsilon_0)$ is small as ε is small.

For the second term in (2.6) we can estimate

$$\begin{aligned}
&\left|\sum_{n=0}^\infty \mathbf{E}_q e^{-\lambda\sigma_n}\psi^\varepsilon_2(q^\varepsilon_{\sigma_n})\right| \leq \sum_{n=0}^\infty \mathbf{E}_q e^{-\lambda\sigma_n}|\psi^\varepsilon_2(q)| \leq \sum_{n=0}^\infty \mathbf{E}_q e^{-\lambda\tau_n}|\psi^\varepsilon_2(q)| \\
&\leq (1 + \frac{M_4}{(\widetilde{u}(\delta))\wedge(-\widetilde{u}(-\delta))})(\widetilde{u}(\delta)-\widetilde{u}(-\delta))^2
\end{aligned}$$

which converges to 0 as $\varepsilon \downarrow 0$. This proves this lemma. □

Lemma 2.6. *We have, for $q \in G(\delta)$, as ε is small, that $|\psi^\varepsilon_2(q)| \leq (\widetilde{u}(\delta)-\widetilde{u}(-\delta))^2$.*

Proof. For the initial point $q \in G(\delta)$ and the time interval $0 \leq t \leq \tau_1$ the trajectory of q^ε_t is traveling in one of the intervals either $[1+\delta', 1+b]$ or $[a-1, -1-\delta']$. Without loss of generality let us assume that $q \in [1+\delta, 1+b]$ and we are traveling in the interval $[1+\delta', 1+b]$. Let $\widetilde{q} = \pi(q)$. Let $B(\widetilde{q}) = b(\widetilde{q}+1)$ and $\Lambda(\widetilde{q}) = \lambda(\widetilde{q}+1)$. Let us extend the function $\Lambda(\bullet)$ to the whole line $\mathbb{R}$. The extended function $\widehat{\Lambda}(\bullet)$ is smooth, bounded, with uniformly bounded derivatives and such that $\widehat{\Lambda}(x) \geq \min\limits_{q\in[1+\delta', 1+b]} \lambda(q)$, $\widehat{\Lambda}(x) = \lambda(1+x)$ for $x \in [\delta', b]$.

Let the process $\widetilde{\widetilde{q}}^\varepsilon_t$ be subject to the stochastic differential equation

$$\dot{\widetilde{\widetilde{q}}}^\varepsilon_t = \frac{B(\widetilde{\widetilde{q}}^\varepsilon_t)}{\widehat{\Lambda}(\widetilde{\widetilde{q}}^\varepsilon_t)+\varepsilon} - \frac{\widehat{\Lambda}'(\widetilde{\widetilde{q}}^\varepsilon_t)}{2(\widehat{\Lambda}(\widetilde{\widetilde{q}}^\varepsilon_t)+\varepsilon)^3} + \frac{1}{\widehat{\Lambda}(\widetilde{\widetilde{q}}^\varepsilon_t)+\varepsilon}\dot{W}_t \ , \ \widetilde{\widetilde{q}}^\varepsilon_0 = \widetilde{q} \ , 0 \leq t < \infty \ .$$

We introduce a stochastic process $\widehat{\widetilde{q}}_t$, $\widehat{\widetilde{q}}_0 = \widetilde{q}$ with generator $\widehat{A}$, subject to the stochastic differential equation

$$\dot{\widehat{\widetilde{q}}}_t = \frac{B(\widehat{\widetilde{q}}_t)}{\widehat{\Lambda}(\widehat{\widetilde{q}}_t)} - \frac{\widehat{\Lambda}'(\widehat{\widetilde{q}}_t)}{2\widehat{\Lambda}^3(\widehat{\widetilde{q}}_t)} + \frac{1}{\widehat{\Lambda}(\widehat{\widetilde{q}}_t)}\dot{W}_t \ , \ \widehat{\widetilde{q}}_0 = \widetilde{q} \ , 0 \le t < \infty \ .$$

Notice that the modified generator $\widehat{A}$ agrees with A before the process $\widetilde{q}_t^{\varepsilon}$ reaches $\widetilde{q}_{\tau_1}^{\varepsilon}$. And before the time τ_1 the process $\widehat{\widetilde{q}}_t^{\varepsilon}$ agrees with the process $\widetilde{q}_t^{\varepsilon}$. Therefore we have,

$$\psi_2^{\varepsilon}(q) = \mathbf{E}_{\widetilde{q}}\left[\int_0^{\tau_1} e^{-\lambda t}[\lambda f(\widehat{\widetilde{q}}_t^{\varepsilon}) - \widehat{A}f(\widehat{\widetilde{q}}_t^{\varepsilon})]dt - e^{-\lambda\tau_1}f(\widehat{\widetilde{q}}_{\tau_1}^{\varepsilon})\right] - f(\widetilde{q}) \ .$$

It is clear by Itô's formula that we have (also see, [10, Section 2]), for the stopping time τ_1,

$$\mathbf{E}_{\widetilde{q}}\left[\int_0^{\tau_1} e^{-\lambda t}[\lambda f(\widehat{\widetilde{q}}_t) - \widehat{A}f(\widehat{\widetilde{q}}_t)]dt - e^{-\lambda\tau_1}f(\widehat{\widetilde{q}}_{\tau_1})\right] - f(\widetilde{q}) = 0 \ .$$

Notice that the function $f \in D \subset D(A)$ is three times continuously differentiable in $[\delta', b]$. This gives the estimate that for some positive $U, V > 0$ and $T = T(\varepsilon)$ we have

$$\begin{aligned}
&|\psi_2^{\varepsilon}(q)| \\
&= \left|\mathbf{E}_{\widetilde{q}}\int_0^{\tau_1} e^{-\lambda t}[\lambda(f(\widehat{\widetilde{q}}_t^{\varepsilon}) - f(\widehat{\widetilde{q}}_t)) - (\widehat{A}f(\widehat{\widetilde{q}}_t^{\varepsilon}) - \widehat{A}f(\widehat{\widetilde{q}}_t))]dt - e^{-\lambda\tau_1}(f(\widehat{\widetilde{q}}_{\tau_1}^{\varepsilon}) - f(\widehat{\widetilde{q}}_{\tau_1}))\right| \\
&\le \mathbf{E}_{\widetilde{q}}\left(\int_0^{T(\varepsilon)} \lambda e^{-\lambda t}dt\,(\mathrm{Lip}(f)) \cdot |\widehat{\widetilde{q}}_t^{\varepsilon} - \widehat{\widetilde{q}}_t| + \right. \\
&\qquad \left.\int_0^{T(\varepsilon)} e^{-\lambda t}dt\,(\mathrm{Lip}(Af)) \cdot |\widehat{\widetilde{q}}_t^{\varepsilon} - \widehat{\widetilde{q}}_t| + (\mathrm{Lip}(f)) \cdot |\widehat{\widetilde{q}}_{\tau_1}^{\varepsilon} - \widehat{\widetilde{q}}_{\tau_1}|\mathbf{1}\,(\tau_1 \le T(\varepsilon))\right) + \\
&\qquad\qquad V\mathbf{P}(\tau_1 \ge T(\varepsilon)) \\
&\le U\left(\max_{0\le t\le T(\varepsilon)} \mathbf{E}_{\widetilde{q}}|\widehat{\widetilde{q}}_t^{\varepsilon} - \widehat{\widetilde{q}}_t|\right) + V\mathbf{P}(\tau_1 \ge T(\varepsilon)) \\
&\le U \max_{0\le t\le T(\varepsilon)} \left(\mathbf{E}_{\widetilde{q}}|\widehat{\widetilde{q}}_t^{\varepsilon} - \widehat{\widetilde{q}}_t|^2\right)^{1/2} + V\mathbf{P}(\tau_1 \ge T(\varepsilon)) \ .
\end{aligned}$$

By the integral form of the stochastic differential equations of the processes $\widehat{\widetilde{q}}_t^{\varepsilon}$ and $\widehat{\widetilde{q}}_t$ we have

$$
\begin{aligned}
&|\widetilde{\widehat{q}}^{\varepsilon}_t - \widehat{\widetilde{q}}_t|^2 \\
&\leq C\left(\left|\int_0^t \left[\left(\frac{B(\widetilde{\widehat{q}}^{\varepsilon}_s)}{\widehat{\Lambda}(\widetilde{\widehat{q}}^{\varepsilon}_s)+\varepsilon} - \frac{\widehat{\Lambda}'(\widetilde{\widehat{q}}^{\varepsilon}_s)}{2(\widehat{\Lambda}(\widetilde{\widehat{q}}^{\varepsilon}_s)+\varepsilon)^3}\right) - \left(\frac{B(\widetilde{\widehat{q}}^{\varepsilon}_s)}{\widehat{\Lambda}(\widetilde{\widehat{q}}^{\varepsilon}_s)} - \frac{\widehat{\Lambda}'(\widetilde{\widehat{q}}^{\varepsilon}_s)}{2(\widehat{\Lambda}(\widetilde{\widehat{q}}^{\varepsilon}_s))^3}\right)\right] ds\right|^2 + \right. \\
&\left|\int_0^t \left[\left(\frac{B(\widetilde{\widehat{q}}^{\varepsilon}_s)}{\widehat{\Lambda}(\widetilde{\widehat{q}}^{\varepsilon}_s)} - \frac{\widehat{\Lambda}'(\widetilde{\widehat{q}}^{\varepsilon}_s)}{2(\widehat{\Lambda}(\widetilde{\widehat{q}}^{\varepsilon}_s))^3}\right) - \left(\frac{B(\widehat{\widetilde{q}}_s)}{\widehat{\Lambda}(\widehat{\widetilde{q}}_s)} - \frac{\widehat{\Lambda}'(\widehat{\widetilde{q}}_s)}{2(\widehat{\Lambda}(\widehat{\widetilde{q}}_s))^3}\right)\right] ds\right|^2 + \\
&\left.\left|\int_0^t \left[\frac{1}{\widehat{\Lambda}(\widetilde{\widehat{q}}^{\varepsilon}_s)+\varepsilon} - \frac{1}{\widehat{\Lambda}(\widetilde{\widehat{q}}^{\varepsilon}_s)}\right] dW_s\right|^2 + \left|\int_0^t \left[\frac{1}{\widehat{\Lambda}(\widetilde{\widehat{q}}^{\varepsilon}_s)} - \frac{1}{\widehat{\Lambda}(\widehat{\widetilde{q}}_s)}\right] dW_s\right|^2\right) .
\end{aligned}
$$

Let $\alpha(\lambda)$ be the Lipschitz constant of $\dfrac{1}{x}$ $(x > \lambda)$, $\beta(\lambda)$ that of $\dfrac{1}{2x^3}$ $(x > \lambda)$, $\gamma(\delta')$ that of $\dfrac{B(\widehat{q})}{\widehat{\Lambda}(q)} - \dfrac{\widehat{\Lambda}'(q)}{2\widehat{\Lambda}(q)^3}$ $(q \geq \delta')$, $\mu(\delta')$ that of $\dfrac{1}{\widehat{\Lambda}(q)}$ $(q \geq \delta')$. Let $m(\delta') \equiv \min_{x\in[\delta',b]} \Lambda(x)$.

We can estimate

$$
\begin{aligned}
&\mathbf{E}_{\widetilde{q}}\left|\int_0^t \left[\left(\frac{B(\widetilde{\widehat{q}}^{\varepsilon}_s)}{\widehat{\Lambda}(\widetilde{\widehat{q}}^{\varepsilon}_s)+\varepsilon} - \frac{\widehat{\Lambda}'(\widetilde{\widehat{q}}^{\varepsilon}_s)}{2(\widehat{\Lambda}(\widetilde{\widehat{q}}^{\varepsilon}_s)+\varepsilon)^3}\right) - \left(\frac{B(\widetilde{\widehat{q}}^{\varepsilon}_s)}{\widehat{\Lambda}(\widetilde{\widehat{q}}^{\varepsilon}_s)} - \frac{\widehat{\Lambda}'(\widetilde{\widehat{q}}^{\varepsilon}_s)}{2(\widehat{\Lambda}(\widetilde{\widehat{q}}^{\varepsilon}_s))^3}\right)\right] ds\right|^2 \\
&\leq A_1(t^2\varepsilon^2[\alpha^2(m(\delta')) + \beta^2(m(\delta'))]) ,
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{E}_{\widetilde{q}}\left|\int_0^t \left[\left(\frac{B(\widetilde{\widehat{q}}^{\varepsilon}_s)}{\widehat{\Lambda}(\widetilde{\widehat{q}}^{\varepsilon}_s)} - \frac{\widehat{\Lambda}'(\widetilde{\widehat{q}}^{\varepsilon}_s)}{2(\widehat{\Lambda}(\widetilde{\widehat{q}}^{\varepsilon}_s))^3}\right) - \left(\frac{B(\widehat{\widetilde{q}}_s)}{\widehat{\Lambda}(\widehat{\widetilde{q}}_s)} - \frac{\widehat{\Lambda}'(\widehat{\widetilde{q}}_s)}{2(\widehat{\Lambda}(\widehat{\widetilde{q}}_s))^3}\right)\right] ds\right|^2 \\
&\leq A_2 t\gamma^2(\delta')\int_0^t \mathbf{E}_{\widetilde{q}}|\widetilde{\widehat{q}}^{\varepsilon}_s - \widehat{\widetilde{q}}_s|^2 ds ,
\end{aligned}
$$

$$
\mathbf{E}_{\widetilde{q}}\left|\int_0^t \left[\frac{1}{\widehat{\Lambda}(\widetilde{\widehat{q}}^{\varepsilon}_s)+\varepsilon} - \frac{1}{\widehat{\Lambda}(\widetilde{\widehat{q}}^{\varepsilon}_s)}\right] dW_s\right|^2 \leq \int_0^t \varepsilon^2\alpha^2(m(\delta'))ds = \varepsilon^2 t\alpha^2(m(\delta')) ,
$$

$$
\mathbf{E}_{\widetilde{q}}\left|\int_0^t \left[\frac{1}{\widehat{\Lambda}(\widetilde{\widehat{q}}^{\varepsilon}_s)} - \frac{1}{\widehat{\Lambda}(\widehat{\widetilde{q}}_s)}\right] dW_s\right|^2 \leq \mu^2(\delta')\int_0^t \mathbf{E}_{\widetilde{q}}|\widetilde{\widehat{q}}^{\varepsilon}_s - \widehat{\widetilde{q}}_s|^2 ds .
$$

We have, by using the above estimates, with a possible change of the constant C, that

$$
\mathbf{E}_{\widetilde{q}}|\widetilde{\widehat{q}}^{\varepsilon}_t - \widehat{\widetilde{q}}_t|^2 \leq C\left(t\varepsilon^2(t(\alpha^2(m(\delta')) + \beta^2(m(\delta'))) + \alpha^2(m(\delta'))) + (t\gamma^2(\delta') + \mu^2(\delta'))\int_0^t \mathbf{E}_{\widetilde{q}}|\widetilde{\widehat{q}}^{\varepsilon}_s - \widehat{\widetilde{q}}_s|^2 ds\right) .
$$

By Bellman-Gronwall inequality we have

$$
\mathbf{E}_{\widetilde{q}}|\widetilde{\widehat{q}}^{\varepsilon}_t - \widehat{\widetilde{q}}_t|^2 \leq Ct\varepsilon^2(t(\alpha^2(m(\delta')) + \beta^2(m(\delta'))) + \alpha^2(m(\delta')))\exp\left(C(t\gamma^2(\delta') + \mu^2(\delta'))t\right) .
$$

As we can check that $|\alpha(m(\delta'))| \leq \frac{1}{m^2(\delta')}$, $\beta(m(\delta')) \leq \frac{A_3}{m^4(\delta')}$, $\gamma(\delta') \leq \frac{A_3}{m^4(\delta')}$ and $|\mu(\delta')| \leq \frac{A_3}{m^2(\delta')}$, this gives, as δ' is small, that

$$\max_{0\leq t\leq T(\varepsilon)} \left(\mathbf{E}_{\widetilde{q}}|\widehat{\widetilde{q}}^{\varepsilon}_t - \widehat{\widetilde{q}}_t|^2\right)^{1/2} \leq$$

$$\leq CT(\varepsilon)\varepsilon(\alpha^2(m(\delta')) + \beta^2(m(\delta')) + \frac{\alpha^2(m(\delta'))}{T(\varepsilon)})^{1/2} \exp\left(C(T(\varepsilon)\gamma^2(\delta') + \mu^2(\delta'))T(\varepsilon)\right)$$

$$\leq CT(\varepsilon)\frac{\varepsilon}{\min\limits_{q\in[1+\delta',1+b]}\lambda^4(q)} \exp\left(CT^2(\varepsilon)\frac{1}{\min\limits_{q\in[1+\delta',1+b]}\lambda^8(q)}\right) .$$

Noticing that by strong Markov property $\mathbf{P}(\tau_1 \geq T(\varepsilon)) \leq K\exp(-pT(\varepsilon))$ for some $p > 0, K > 0$, we see that

$$|\psi_2^{\varepsilon}(q)| \leq CT(\varepsilon)\frac{\varepsilon}{\min\limits_{q\in[1+\delta',1+b]}\lambda^4(q)} \exp\left(CT^2(\varepsilon)\frac{1}{\min\limits_{q\in[1+\delta',1+b]}\lambda^8(q)}\right) + V\exp(-pT(\varepsilon)) .$$

Let us choose $T(\varepsilon) = \sqrt{\ln\ln\frac{1}{\varepsilon}}$. We will then have

$$|\psi_2^{\varepsilon}(q)| \leq C\left(\ln\ln\frac{1}{\varepsilon}\right)^{1/2}\frac{\varepsilon}{\min\limits_{q\in[1+\delta',1+b]}\lambda^4(q)}\left(\ln\frac{1}{\varepsilon}\right)^{\frac{C}{\min\limits_{q\in[1+\delta',1+b]}\lambda^8(q)}} + V\exp(-p\sqrt{\ln\ln\frac{1}{\varepsilon}})) .$$

For fixed $\delta' > 0$, one can choose ε small enough such that

$$|\psi_2^{\varepsilon}(q)| \leq \frac{U_0\varepsilon^{\kappa}}{\min\limits_{q\in[1+\delta',1+b]\cup[-1+a,-1-\delta']}\lambda^4(q)} + U_0\exp(-p\sqrt{\ln\ln\frac{1}{\varepsilon}})$$

for some $U_0 > 0$, $p > 0$ and $0 < \kappa < 1$. As we choose first $\varepsilon \downarrow 0$ and then $\delta' \downarrow 0$, this gives that as ε is small we have $|\psi_2^{\varepsilon}(q)| \leq (\widetilde{u}(\delta) - \widetilde{u}(-\delta))^2$. $\square$

Lemma 2.7. *We have, as $\varepsilon, \delta, \delta'$ are small, for $q \in [-1-\delta', 1+\delta']$ and $C > 0$, that*

$$\left|\mathbf{P}_q(\pi(q^{\varepsilon}_{\sigma_0}) = \delta) - \frac{\widetilde{u}(0) - \widetilde{u}(-\delta)}{\widetilde{u}(\delta) - \widetilde{u}(-\delta)}\right| \leq \frac{\widetilde{u}(\delta') - \widetilde{u}(0) + C\varepsilon}{\widetilde{u}(\delta) - \widetilde{u}(-\delta)} ,$$

$$\left|\mathbf{P}_q(\pi(q^{\varepsilon}_{\sigma_0}) = -\delta) - \frac{\widetilde{u}(\delta) - \widetilde{u}(0)}{\widetilde{u}(\delta) - \widetilde{u}(-\delta)}\right| \leq \frac{\widetilde{u}(\delta') - \widetilde{u}(0) + C\varepsilon}{\widetilde{u}(\delta) - \widetilde{u}(-\delta)} .$$

Proof. Let $\widetilde{q} = \pi(q) \in [-\delta', \delta']$. We have, for bounded positive functions $C_1(\delta, \varepsilon)$, $C_2(\delta, \varepsilon)$ and positive constants C_1, C_2, C, that

$$\begin{aligned}
&\left|\mathbf{P}_q(\pi(q^\varepsilon_{\sigma_0}) = \delta) - \frac{\widetilde{u}(0) - \widetilde{u}(-\delta)}{\widetilde{u}(\delta) - \widetilde{u}(-\delta)}\right| \\
&= \left|\frac{u^\varepsilon(q) - u^\varepsilon(-1-\delta)}{u^\varepsilon(1+\delta) - u^\varepsilon(-1-\delta)} - \frac{\widetilde{u}(0) - \widetilde{u}(-\delta)}{\widetilde{u}(\delta) - \widetilde{u}(-\delta)}\right| \\
&= \left|\frac{\widetilde{u}(0) - \widetilde{u}(-\delta) + \widetilde{u}(\widetilde{q}) - \widetilde{u}(0) + C_1(\delta, \varepsilon)\varepsilon}{\widetilde{u}(\delta) - \widetilde{u}(-\delta) + C_2(\delta, \varepsilon)\varepsilon} - \frac{\widetilde{u}(0) - \widetilde{u}(-\delta)}{\widetilde{u}(\delta) - \widetilde{u}(-\delta)}\right| \\
&\leq \frac{(\widetilde{u}(\widetilde{q}) - \widetilde{u}(0) + C_1\varepsilon)(\widetilde{u}(\delta) - \widetilde{u}(-\delta)) + C_2\varepsilon(\widetilde{u}(0) - \widetilde{u}(-\delta))}{(\widetilde{u}(\delta) - \widetilde{u}(-\delta))^2} \\
&\leq \frac{\widetilde{u}(\delta') - \widetilde{u}(0) + C\varepsilon}{\widetilde{u}(\delta) - \widetilde{u}(-\delta)} \ .
\end{aligned}$$

The estimate of $\mathbf{P}_q(\pi(q^\varepsilon_{\sigma_0}) = -\delta)$ is similar. $\square$

Lemma 2.8. *We have, as ε are small, for $q \in [-1-\delta', 1+\delta']$, that $|(I)^\varepsilon(q)| \leq C(\widetilde{u}(\delta) - \widetilde{u}(-\delta))^2$.*

Proof. We have, using Lemma 2.7, that

$$\begin{aligned}
&|(I)^\varepsilon(q)| \\
&= |\mathbf{E}_q f(\pi(q^\varepsilon_{\sigma_0})) - f(\pi(q))| \\
&= |(f(\delta) - f(0))\mathbf{P}_q(\pi(q^\varepsilon_{\sigma_0}) = \delta) - (f(0) - f(-\delta))\mathbf{P}_q(\pi(q^\varepsilon_{\sigma_0}) = -\delta) + (f(0) - f(\pi(q)))| \\
&\leq \left|(f(\delta) - f(0))\frac{\widetilde{u}(0) - \widetilde{u}(-\delta)}{\widetilde{u}(\delta) - \widetilde{u}(-\delta)} - (f(0) - f(-\delta))\frac{\widetilde{u}(\delta) - \widetilde{u}(0)}{\widetilde{u}(\delta) - \widetilde{u}(-\delta)}\right| + \\
&\quad C_4\frac{\widetilde{u}(\delta') - \widetilde{u}(0) + M\varepsilon}{\widetilde{u}(\delta) - \widetilde{u}(-\delta)} + C_5(\widetilde{u}(\delta') - \widetilde{u}(0)) \\
&= \left|\frac{(\widetilde{u}(0) - \widetilde{u}(-\delta))(\widetilde{u}(\delta) - \widetilde{u}(0))}{\widetilde{u}(\delta) - \widetilde{u}(-\delta)}\left(\frac{f(\delta) - f(0)}{\widetilde{u}(\delta) - \widetilde{u}(0)} - \frac{f(0) - f(-\delta)}{\widetilde{u}(0) - \widetilde{u}(-\delta)}\right)\right| + \\
&\quad C_4\frac{\widetilde{u}(\delta') - \widetilde{u}(0) + M\varepsilon}{\widetilde{u}(\delta) - \widetilde{u}(-\delta)} + C_5(\widetilde{u}(\delta') - \widetilde{u}(0)) \\
&\leq C_3(\widetilde{u}(\delta) - \widetilde{u}(-\delta))^2 + C_4\frac{\widetilde{u}(\delta') - \widetilde{u}(0) + M\varepsilon}{\widetilde{u}(\delta) - \widetilde{u}(-\delta)} + C_5(\widetilde{u}(\delta') - \widetilde{u}(0)) \ .
\end{aligned}$$

We have used our gluing condition $D^+_{\widetilde{u}} f(0) = D^-_{\widetilde{u}} f(0)$. Now we choose first $\varepsilon \downarrow 0$ then $\delta' \downarrow 0$, we get, as ε is small, that $|(I)^\varepsilon(q)| \leq C(\widetilde{u}(\delta) - \widetilde{u}(-\delta))^2$. $\square$

Lemma 2.9. *As $\varepsilon, \delta, \delta'$ are small, for $q \in [-1-\delta', 1+\delta']$ we have,*

$$\mathbf{E}_q\sigma_0 \leq C(\widetilde{u}(\delta) - \widetilde{u}(-\delta))(\widetilde{v}(\delta) - \widetilde{v}(-\delta)) \ , \ \mathbf{E}_q(1 - e^{-\lambda\sigma_0}) \leq C(\widetilde{u}(\delta) - \widetilde{u}(-\delta))(\widetilde{v}(\delta) - \widetilde{v}(-\delta)) \ .$$

Proof. We apply the well known formula for the expected exit time (see, for example [14, Chapter VII, Theorem 3.6]) and we have

$$\mathbf{E}_q\sigma_0 = \int_{-1-\delta}^{1+\delta} G^\varepsilon(q,r)dv^\varepsilon(r) \ ,$$

where the Green function

$$G^\varepsilon(q,r) = \begin{cases} \dfrac{(u^\varepsilon(q)-u^\varepsilon(-1-\delta))(u^\varepsilon(1+\delta)-u^\varepsilon(r))}{u^\varepsilon(1+\delta)-u^\varepsilon(-1-\delta)} \text{ for } -1-\delta\le q\le r\le 1+\delta \ , \\ \dfrac{(u^\varepsilon(r)-u^\varepsilon(-1-\delta))(u^\varepsilon(1+\delta)-u^\varepsilon(q))}{u^\varepsilon(1+\delta)-u^\varepsilon(-1-\delta)} \text{ for } -1-\delta\le r\le q\le 1+\delta \ , \\ 0 \text{ otherwise } \ . \end{cases}$$

Therefore it is easy to estimate

$$\begin{aligned} &\mathbf{E}_q\sigma_0 \\ &\le (u^\varepsilon(1+\delta)-u^\varepsilon(-1-\delta))(v^\varepsilon(1+\delta)-v^\varepsilon(-1-\delta)) \\ &\le (\widetilde{u}(\delta)-\widetilde{u}(-\delta)+C_6\varepsilon)(\widetilde{v}(\delta)-\widetilde{v}(-\delta)+C_7\varepsilon) \\ &\le C(\widetilde{u}(\delta)-\widetilde{u}(-\delta))(\widetilde{v}(\delta)-\widetilde{v}(-\delta)) \end{aligned}$$

as desired.

This helps us to find

$$\mathbf{E}_q(1-e^{-\lambda\sigma_0}) = \lambda\mathbf{E}_q\left[\int_0^{\sigma_0} e^{-\lambda s}ds\right] \le \lambda\mathbf{E}_q\sigma_0 \le C(\widetilde{u}(\delta)-\widetilde{u}(-\delta))(\widetilde{v}(\delta)-\widetilde{v}(-\delta)) \ .$$

□

Lemma 2.10. *For $q\in G(\delta)$ and δ sufficiently small, we have*

$$\lim_{\delta'\downarrow 0}\lim_{\varepsilon\downarrow 0}\mathbf{E}_q e^{-\lambda\tau_1} \le 1 - C(\widetilde{u}(\delta))\wedge(-\widetilde{u}(-\delta)) \ .$$

Proof. Without loss of generality let $q\in[1+\delta,1+b]$. The expected value $M^\varepsilon(q)=\mathbf{E}_q e^{-\lambda\tau_1}$ is the solution of the differential equation $D_{v^\varepsilon}D_{u^\varepsilon}M^\varepsilon(q)=\lambda M^\varepsilon(q)$, $M^\varepsilon(1+\delta')=M^\varepsilon(1+b)=1$.

There exist two solutions $f_1^\lambda(q)$, $f_2^\lambda(q)$ of the equation $D_vD_uf=\lambda f$ with $f_1^\lambda(1)=f_2^\lambda(1+b)=1$ and $f_1^\lambda(1+b)=f_2^\lambda(1)=0$. The derivatives $D_uf_1^\lambda(x)$, $D_uf_2^\lambda(x)$ are increasing functions, $-\infty<\lim\limits_{q\downarrow 1}D_u(f_1^\lambda+f_2^\lambda)(q)<0$, $0<\lim\limits_{q\uparrow 1+b}D_u(f_1^\lambda+f_2^\lambda)(q)<\infty$ (see [4], [11]).

We shall make use of Lemma 2.6. Since $q\in[1+\delta,1+b]$ we see that $\sigma_0=0$. Lemma 2.6 tells us that, for $k=1,2$, we have

$$\lim_{\varepsilon\downarrow 0}\left|\mathbf{E}_q\left[\int_0^{\tau_1} e^{-\lambda t}[\lambda f_k^\lambda(q_t^\varepsilon) - D_v D_u f_k^\lambda(q_t^\varepsilon)]dt + e^{-\lambda\tau_1} f_k^\lambda(q_{\tau_1}^\varepsilon)\right] - f_k^\lambda(q)\right| \leq (\widetilde{u}(\delta) - \widetilde{u}(-\delta))^2 \ .$$

Taking into account the definitions of f_1^λ, f_2^λ we see that the above inequality gives

$$\left|\lim_{\varepsilon\downarrow 0}\mathbf{E}_q e^{-\lambda\tau_1} f_k^\lambda(q_{\tau_1}^\varepsilon) - f_k^\lambda(q)\right| \leq (\widetilde{u}(\delta) - \widetilde{u}(-\delta))^2 \ .$$

Since $f_k^\lambda(q_{\tau_1}^\varepsilon) = f_k^\lambda(1+\delta')$ when $q_{\tau_1}^\varepsilon = 1+\delta'$ and $f_k^\lambda(q_{\tau_1}^\varepsilon) = f_k^\lambda(1+b)$ when $q_{\tau_1}^\varepsilon = 1+b$, we see that for some $K>0$ we have

$$\left|\lim_{\varepsilon\downarrow 0}\mathbf{E}_q e^{-\lambda\tau_1} - \frac{(f_2^\lambda(1+b) - f_2^\lambda(1+\delta'))f_1^\lambda(q) + (f_1^\lambda(1+\delta') - f_1^\lambda(1+b))f_2^\lambda(q)}{f_1^\lambda(1+\delta')f_2^\lambda(1+b) - f_1^\lambda(1+b)f_2^\lambda(1+\delta')}\right| \leq K(\widetilde{u}(\delta) - \widetilde{u}(-\delta))^2 \ .$$

(The expression

$$\frac{(f_2^\lambda(1+b) - f_2^\lambda(1+\delta'))f_1^\lambda(q) + (f_1^\lambda(1+\delta') - f_1^\lambda(1+b))f_2^\lambda(q)}{f_1^\lambda(1+\delta')f_2^\lambda(1+b) - f_1^\lambda(1+b)f_2^\lambda(1+\delta')}$$

is the solution of the equation $\lambda f(q) = D_v D_u f$ with $f(1+\delta') = f(1+b) = 1$.)

This gives

$$\left|\lim_{\delta'\downarrow 0}\lim_{\varepsilon\downarrow 0}\mathbf{E}_q(1 - e^{-\lambda\tau_1}) - [1 - (f_1^\lambda(q) + f_2^\lambda(q))]\right| \leq K(\widetilde{u}(\delta) - \widetilde{u}(-\delta))^2 \ .$$

Taking into account that $-\infty < \lim\limits_{q\downarrow 1} D_u(f_1^\lambda + f_2^\lambda)(q) < 0$, $0 < \lim\limits_{q\uparrow 1+b} D_u(f_1^\lambda + f_2^\lambda)(q) < \infty$ we see from the above estimate that

$$\lim_{\delta'\downarrow 0}\lim_{\varepsilon\downarrow 0}\mathbf{E}_q(1 - e^{-\lambda\tau_1}) \geq C(\widetilde{u}(\delta))$$

for $q \in [1+\delta, 1+b]$ and δ sufficiently small. The case of $\widetilde{u}(-\delta)$ is handled in a similar way. $\square$

3 A two dimensional model problem

In this section we discuss a two dimensional model problem. We work with a Smoluchowski-Kramers approximation in the plane $\mathbb{R}^2$. Let us suppose that the friction coefficient $\lambda(\bullet)$ depends on the y variable only: $\lambda(x, y) = \lambda(y)$. Suppose for $y \in [-1, 1]$ we have $\lambda(y) = 0$. For $y \notin [-1, 1]$ we have $\lambda(y) > 0$. For simplicity of presentation we also assume that the drift is zero: $\boldsymbol{b}(\bullet) = \mathbf{0}$. All the other assumptions about $\lambda(\bullet)$ are the same as was made in Section 1.

In addition, we assume that for $\varepsilon > 0$,

$$\int_{-\varepsilon-1}^{-1} \frac{1}{\lambda(y)} dy = \int_{1}^{1+\varepsilon} \frac{1}{\lambda(y)} dy = \infty \ .$$

(In the case that both integrals converge the proof of Lemma 3.1 repeat that in the case of both integrals divergent but we do not know anything about the case of one integral convergent and the other divergent.)

As we already introduced in equation (1.8) of Section 1, we are actually considering the stochastic differential equation for the position of the particle $\boldsymbol{q}_t^\varepsilon \in \mathbb{R}^2$ as follows:

$$\dot{\boldsymbol{q}}_t^\varepsilon = -\frac{\nabla\lambda(\boldsymbol{q}_t^\varepsilon)}{2(\lambda(\boldsymbol{q}_t^\varepsilon)+\varepsilon)^3} + \frac{1}{\lambda(\boldsymbol{q}_t^\varepsilon)+\varepsilon}\dot{\boldsymbol{W}}_t \ , \ \boldsymbol{q}_0^\varepsilon = \boldsymbol{q}_0 \in \mathbb{R}^2 \ , \ \varepsilon > 0 \ . \tag{3.1}$$

By taking into account our assumption on the friction coefficient λ we can write the above equation in coordinate form. Let $\boldsymbol{q}_t^\varepsilon = (x_t^\varepsilon, y_t^\varepsilon)$. Let $\boldsymbol{W}_t = (W_t^1, W_t^2)$. We have

$$\begin{cases} \dot{x}_t^\varepsilon = \dfrac{1}{\lambda(y_t^\varepsilon)+\varepsilon}\dot{W}_t^1 \ , \ x_0^\varepsilon = x_0 \in \mathbb{R} \ , \\ \dot{y}_t^\varepsilon = -\dfrac{\lambda'(y_t^\varepsilon)}{2(\lambda(y_t^\varepsilon)+\varepsilon)^3} + \dfrac{1}{\lambda(y_t^\varepsilon)+\varepsilon}\dot{W}_t^2 \ , \ y_0^\varepsilon = y_0 \in \mathbb{R} \ . \end{cases} \tag{3.2}$$

Let $a < 0 < b$ be given. Throughout this section we will assume that our process $\boldsymbol{q}_t^\varepsilon$ is stopped once it exits from the domain $\{(x,y) \in \mathbb{R}^2 : a-1 \leq y \leq b+1\}$. We therefore suppose that $y_0 \in [a-1, b+1]$.

Note that, similarly as in Section 2, the process y_t^ε is a strong Markov process subject to a generalized second order differential operator in the form $D_{v^\varepsilon(y)}D_{u^\varepsilon(y)}$ where

$$u^\varepsilon(y) = \int_0^y (\lambda(s)+\varepsilon)ds \ , \ v^\varepsilon(y) = 2\int_0^y (\lambda(s)+\varepsilon)ds \ . \tag{3.3}$$

Let

$$u(y) = \int_0^y \lambda(s)ds \ , \ v(y) = 2\int_0^y \lambda(s)ds \ . \tag{3.4}$$

We have the obvious relation $u^\varepsilon(y) = u(y) + \varepsilon y$ and $v^\varepsilon(y) = v(y) + 2\varepsilon y$.

Let us identify points in the x direction $x \sim x + 2\pi$. Therefore we get a process on the cylinder $S^1 \times [a-1, b+1]$, stopped once it hits the boundary $\{y = a-1 \text{ or } b+1\}$. Let

$$\begin{cases} \theta_t^\varepsilon = x_t^\varepsilon \mod 2\pi \ , \\ y_t^\varepsilon = y_t^\varepsilon \ . \end{cases}$$

In the rest of this section we refer to the process $\boldsymbol{q}_t^\varepsilon$ as the one on a cylinder: $\boldsymbol{q}_t^\varepsilon = (\theta_t^\varepsilon, y_t^\varepsilon)$ is on the cylinder $S^1 \times [a-1, b+1]$. When we speak about the process

$\boldsymbol{q}_t^\varepsilon$ on the domain $\{(x,y) \in \mathbb{R}^2 : a-1 \le y \le b+1\} \subset \mathbb{R}^2$ we will instead refer to the coordinate representation $(x_t^\varepsilon, y_t^\varepsilon)$.

Let $\mathfrak{C}$ be the product $S^1 \times [a,b]$ with all points $S^1 \times \{0\}$ identified, forming the point $\mathfrak{o}$. A generic point on $\mathfrak{C}$ will be denoted $\widetilde{\boldsymbol{q}} = (\theta, \widetilde{y})$ where $\theta \in S^1$ and $\widetilde{y} \in [a,b]$. All points $(\theta, 0)$ correspond to $\mathfrak{o}$.

Let us consider the following projection map $\boldsymbol{\pi} : S^1 \times [a-1, b+1] \to \mathfrak{C}$. We let

$$\boldsymbol{\pi}(\theta, y) = \begin{cases} (\theta, y-1) \ , & \text{for } 1 < y \le b+1 \ ; \\ (\theta, y+1) \ , & \text{for } a-1 \le y < -1 \ ; \\ \mathfrak{o} \ , \quad \text{for } -1 \le y \le 1 \ . \end{cases} \tag{3.5}$$

Let $\boldsymbol{\pi}(\boldsymbol{q}_t^\varepsilon) = \widetilde{\boldsymbol{q}}_t^\varepsilon = (\theta_t^\varepsilon, \widetilde{y}_t^\varepsilon)$. We see that $\widetilde{y}_t^\varepsilon = \pi(y_t^\varepsilon)$ where π is the projection map introduced in Section 2.

Let, as in Section 2, $\widetilde{u}(\widetilde{y}) = u(\widetilde{y}-1)$ for $\widetilde{y} < 0$ and $\widetilde{u}(\widetilde{y}) = u(\widetilde{y}+1)$ for $\widetilde{y} > 0$ and $\widetilde{u}(0) = u(1) = u(-1)$; $\widetilde{v}(\widetilde{y}) = v(\widetilde{y}-1)$ for $\widetilde{y} < 0$ and $\widetilde{v}(\widetilde{y}) = v(\widetilde{y}+1)$ for $\widetilde{y} > 0$ and $\widetilde{v}(0) = \widetilde{v}(1) = \widetilde{v}(-1)$. The functions $\widetilde{u}(\widetilde{y})$ and $\widetilde{v}(\widetilde{y})$ are continuous strictly increasing functions on $[a,b]$. Let $\widetilde{\lambda}(\widetilde{y}) = \lambda(\widetilde{y}-1)$ for $\widetilde{y} < 0$ and $\widetilde{\lambda}(\widetilde{y}) = \lambda(\widetilde{y}+1)$ for $\widetilde{y} > 0$ and $\widetilde{\lambda}(0) = 0$.

Let A be the operator given, for $\widetilde{y} \neq 0$, by the formula

$$Af(\theta, \widetilde{y}) = D_{\widetilde{u}(\widetilde{y})} D_{\widetilde{v}(\widetilde{y})} f + \frac{1}{\widetilde{\lambda}^2(\widetilde{y})} \frac{\partial^2}{\partial \theta^2} f \ . \tag{3.6}$$

Let $D(A)$ be the subset of the space $\mathbf{C}(\mathfrak{C})$ consisting of functions $f(\widetilde{\boldsymbol{q}})$ for which $Af(\theta, \widetilde{y})$ is defined and continuous for $\widetilde{y} \neq 0$, the derivatives in it being continuous; such that finite limits

$$\lim_{\theta' \to \theta, \widetilde{y} \to 0-} D_{\widetilde{u}(\widetilde{y})} f(\theta', \widetilde{y}) \ , \quad \lim_{\theta' \to \theta, \widetilde{y} \to 0+} D_{\widetilde{u}(\widetilde{y})} f(\theta', \widetilde{y}) \ , \tag{3.7}$$

exist;

$$\lim_{\theta' \to \theta, \widetilde{y} \to 0} Af(\theta', \widetilde{y}) \tag{3.8}$$

exists and does not depend on θ;

$$\lim_{\theta' \to \theta, \widetilde{y} \to a} Af(\theta', \widetilde{y}) = \lim_{\theta' \to \theta, \widetilde{y} \to b} Af(\theta', \widetilde{y}) = 0 \ ; \tag{3.9}$$

and

$$\int_0^{2\pi} \lim_{\theta' \to \theta, \widetilde{y} \to 0-} D_{\widetilde{u}(\widetilde{y})} f(\theta', \widetilde{y}) d\theta = \int_0^{2\pi} \lim_{\theta' \to \theta, \widetilde{y} \to 0+} D_{\widetilde{u}(\widetilde{y})} f(\theta', \widetilde{y}) d\theta \ . \tag{3.10}$$

It is worth mentioning here that the above condition (3.10) in the definition of $D(A)$ can be replaced by the condition that $\lim\limits_{\theta' \to \theta, \widetilde{y} \to 0-} D_{\widetilde{u}(\widetilde{y})} f(\theta', \widetilde{y})$ and $\lim\limits_{\theta' \to \theta, \widetilde{y} \to 0+} D_{\widetilde{u}(\widetilde{y})} f(\theta', \widetilde{y})$ not depending on θ and coinciding. In this case the proof of Lemma 3.1 remains the same.

Let us define, for $f \in D(A)$, $Af(\theta, a)$ and $Af(\theta, b)$ as the limits (3.9) and $Af(\mathfrak{o})$ as the limit (3.8). The operator A defined on $D(A)$ is a linear operator $D(A) \mapsto \mathbf{C}(\mathfrak{C})$.

Lemma 3.1. *The closure $\overline{A|_{D(A)}}$ of the operator $A|_{D(A)}$ exists and is the infinitesimal operator of a Markov semigroup on $\mathbf{C}(\mathfrak{C})$.*

(The corresponding Markov process $\widetilde{\boldsymbol{q}}_t$ stops after reaching the boundary of $\mathfrak{C}$ ($\widetilde{y} = a$ or b).)

Proof. We use the Hille-Yosida theorem and we check the following:

• The domain $D(A)$ is dense in $\mathbf{C}(\mathfrak{C})$.

This is because we can approximate every function g in $\mathbf{C}(\mathfrak{C})$ by a function f which is smooth, close to g outside a neighborhood of $\mathfrak{o}$ and is equal to $g(\mathfrak{o})$ in the neighborhood of $\mathfrak{o}$. This function f satisfies our restrictions on $D(A)$ and can approximate the function g with respect to the norm of $\mathbf{C}(\mathfrak{C})$ as we choose the neighborhood of $\mathfrak{o}$ small enough.

• The operator $A|_{D(A)}$ satisfies the maximum principle: for $f \in D(A)$, if this function reaches its maximum value at a point $\widetilde{\boldsymbol{q}} \in \mathfrak{C}$ we have $Af(\widetilde{\boldsymbol{q}}) \leq 0$.

Indeed, for $\widetilde{\boldsymbol{q}} = (\theta, a)$ or (θ, b), we have $Af(\widetilde{\boldsymbol{q}}) = 0$. If $\widetilde{\boldsymbol{q}} = (\theta, \widetilde{y})$, $\widetilde{y} \neq 0$ the first partial derivatives at $\widetilde{\boldsymbol{q}}$ are equal to 0 and $\dfrac{\partial^2}{\partial \theta^2} f(\theta, \widetilde{y}) \leq 0$, $D_{\widetilde{v}(\widetilde{y})} D_{\widetilde{u}(\widetilde{y})} \leq 0$. Finally, if $\widetilde{\boldsymbol{q}} = \mathfrak{o}$ we have the left-hand derivative $D^-_{\widetilde{u}(\widetilde{y})} f(\theta, 0) \geq 0$, the right-hand derivative $D^+_{\widetilde{u}(\widetilde{y})} f(\theta, 0) \leq 0$ and by (3.10) both these derivatives are equal to 0. It follows then that the limit as $\widetilde{y} \to 0$ of the second $\widetilde{y}$-derivative is non-positive for all $\theta \in S^1$. Since the integral over S^1 of the second θ derivative is equal to 0 for all $\widetilde{y} \neq 0$, taking into account that $Af(\mathfrak{o})$ is equal to the limit (3.8), we have that $Af(\mathfrak{o}) \leq 0$.

It follows from the maximum principle that for $\lambda > 0$ the operator $\lambda I - A|_{D(A)}$ does not send to zero any function that is not equal 0, and this linear operator has an inverse (that is not defined on the whole $\mathbf{C}(\mathfrak{C})$), with $\|(\lambda I - A|_{D(A)})^{-1}\| \leq \lambda^{-1}$. Every bounded linear operator does have a closure (which is just its extension by continuity), and with it the operators $\lambda I - A|_{D(A)}$ and $A|_{D(A)}$ also have closures.

• Finally, to check that we can apply Hille-Yosida theorem to the closure $\overline{A|_{D(A)}}$ we have only to check that the bounded operator $(\lambda I - A|_{D(A)})^{-1}$ is defined on a dense set. That is, for a dense subset of $F \in \mathbf{C}(\mathfrak{C})$ there exists a solution $f \in D(A)$ of the equation

$$\lambda f - Af = F \,. \tag{3.11}$$

Let us take $F(\theta, \widetilde{y}) = e^{in\theta} G(\widetilde{y})$, defining $F(\mathfrak{o})$ as its limit as $\widetilde{y} \to 0$. Of course for $n \neq 0$ we have to have $\lim\limits_{\widetilde{y} \to 0} G(\widetilde{y})$ (which limit we'll take as the value $G(0)$) equal to 0.

We shall look for the solution $f \in D(A)$ of the equation (3.11) in the form $f(\theta, \widetilde{y}) = e^{in\theta} g(\widetilde{y})$ (again, for $n \neq 0$ it should be $g(0) = \lim\limits_{\widetilde{y} \to 0} g(\widetilde{y}) = 0$).

The differential equation for $g(\widetilde{y})$ following from (3.11) is the ordinary differential equation

$$(\lambda + \frac{n^2}{\widetilde{\lambda}^2(\widetilde{y})})g(\widetilde{y}) - D_{\widetilde{v}(\widetilde{y})}D_{\widetilde{u}(\widetilde{y})}g(\widetilde{y}) = G(\widetilde{y}) \ , \tag{3.12}$$

and it should be solved with the boundary conditions $\frac{n^2}{\widetilde{\lambda}^2(a)}g(a) - D_{\widetilde{v}(\widetilde{y})}D_{\widetilde{u}(\widetilde{y})}g(a) = \frac{n^2}{\widetilde{\lambda}^2(b)}g(b) - D_{\widetilde{v}(\widetilde{y})}D_{\widetilde{u}(\widetilde{y})}g(b) = 0$, $D^-_{\widetilde{u}(\widetilde{y})}g(0) = D^+_{\widetilde{u}(\widetilde{y})}g(0)$ and for $n \neq 0$, $g(0) = 0$. From the boundary conditions we get at once $g(a) = \lambda^{-1}G(a)$ and $g(b) = \lambda^{-1}G(b)$.

For $n = 0$ the equation (3.12) with the boundary conditions $D_{\widetilde{u}(\widetilde{y})}D_{\widetilde{v}(\widetilde{y})}g(a) = D_{\widetilde{u}(\widetilde{y})}D_{\widetilde{v}(\widetilde{y})}g(b) = 0$ and the gluing condition $D^-_{\widetilde{u}(\widetilde{y})}g(0) = D^+_{\widetilde{u}(\widetilde{y})}g(0)$ is just the ordinary differential equation for a one-dimensional diffusion process that has been considered infinitely many times, and it has a solution for every $G \in \mathbf{C}[a,b]$. Let us go to the case $n \neq 0$. We are going to consider the intervals $[a, 0)$ and $(0, b]$ separately; what follows is about the interval $(0, b]$.

Similarly to how it is done in, e.g.[4], we can prove that there exist two non-negative solutions $\xi_1(\widetilde{y})$ and $\xi_2(\widetilde{y})$ of the equation

$$(\lambda + \frac{n^2}{\widetilde{\lambda}^2(\widetilde{y})})\xi_i(\widetilde{y}) - D_{\widetilde{v}(\widetilde{y})}D_{\widetilde{u}(\widetilde{y})}\xi_i(\widetilde{y}) = 0 \ , \ 0 < \widetilde{y} \leq b \ , \tag{3.13}$$

the first one increasing and the second one decreasing, $\xi_1(0) = \xi_2(b) = 0$, $\xi_1(b) < \infty$, $\xi_2(0+) = \infty$. The derivatives $D_{\widetilde{u}(\widetilde{y})}\xi_i(\widetilde{y})$ are increasing, $D_{\widetilde{u}(\widetilde{y})}\xi_1(0) = 0$, $D_{\widetilde{u}(\widetilde{y})}\xi_2(b) < 0$.

It is easily checked that the Wronskian

$$W(\widetilde{y}) = \det \begin{pmatrix} D_{\widetilde{u}(\widetilde{y})}\xi_1(\widetilde{y}) & D_{\widetilde{u}(\widetilde{y})}\xi_2(\widetilde{y}) \\ \xi_1(\widetilde{y}) & \xi_2(\widetilde{y}) \end{pmatrix}$$

(both summands $D_{\widetilde{u}(\widetilde{y})}\xi_1(\widetilde{y}) \cdot \xi_2(\widetilde{y})$ and $-D_{\widetilde{u}(\widetilde{y})}\xi_2(\widetilde{y}) \cdot \xi_1(\widetilde{y})$ are positive) does not depend on $\widetilde{y}$: $W(\widetilde{y}) \equiv W > 0$.

Now we define, for $\widetilde{y} \in [0, b]$,

$$\widetilde{g}(\widetilde{y}) = \frac{1}{W}\left[\xi_2(\widetilde{y})\int_0^{\widetilde{y}} \xi_1(z) \cdot G(z)d\widetilde{v}(z) + \xi_1(\widetilde{y})\int_{\widetilde{y}}^b \xi_2(z) \cdot G(z)d\widetilde{v}(z)\right] . \tag{3.14}$$

It is easily checked that $\lambda\widetilde{g}(\widetilde{y}) - A\widetilde{g}(\widetilde{y}) = G(\widetilde{y})$ for $0 < \widetilde{y} \leq b$.

Of course

$$|\widetilde{g}(\widetilde{y})| \leq \frac{\|G\|}{W}\left[\xi_2(\widetilde{y})\int_0^{\widetilde{y}} \xi_1(z)d\widetilde{v}(z) + \xi_1(\widetilde{y})\int_{\widetilde{y}}^b \xi_2(z)d\widetilde{v}(z)\right] . \tag{3.15}$$

Let us check that this goes to 0 as $y \to 0+$.

We have:

$$\xi_i(z) = \frac{D_{\widetilde{v}(\widetilde{y})}D_{\widetilde{u}(\widetilde{y})}\xi_i(z)}{\lambda + n^2/\widetilde{\lambda}^2(z)}$$

so the first summand in the brackets in (3.15) is less or equal

$$\xi_2(\widetilde{y}) \cdot \frac{D_{\widetilde{u}(\widetilde{y})}\xi_1(\widetilde{y}) - D_{\widetilde{u}(\widetilde{y})}\xi_1(0)}{\min\limits_{0\le z\le\widetilde{y}}[\lambda + n^2/\widetilde{\lambda}^2(z)]} = \frac{\xi_2(\widetilde{y}) \cdot D_{\widetilde{u}(\widetilde{y})}\xi_1(\widetilde{y})}{\min\limits_{0\le z\le\widetilde{y}}[\lambda + n^2/\widetilde{\lambda}^2(z)]} < \frac{W}{\min\limits_{0\le z\le\widetilde{y}}[\lambda + n^2/\widetilde{\lambda}^2(z)]} ,$$

and it goes to zero as $\widetilde{y} \to 0+$.

The second summand in (3.15) is less or equal

$$\xi_1(\widetilde{y}) \cdot \frac{D_{\widetilde{u}(\widetilde{y})}\xi_2(c) - D_{\widetilde{u}(\widetilde{y})}\xi_2(\widetilde{y})}{\min\limits_{\widetilde{y}\le z\le c}[\lambda + n^2/\widetilde{\lambda}^2(z)]} + \xi_1(\widetilde{y}) \cdot \frac{D_{\widetilde{u}(\widetilde{y})}\xi_2(b) - D_{\widetilde{u}(\widetilde{y})}\xi_2(c)}{\min\limits_{c\le z\le b}[\lambda + n^2/\widetilde{\lambda}^2(z)]} , \tag{3.16}$$

where $\widetilde{y} < c < b$. The first term in (3.16) is less or equal

$$\frac{-\xi_1(y) \cdot D_{\widetilde{u}(\widetilde{y})}\xi_2(\widetilde{y})}{\min\limits_{\widetilde{y}\le z\le c}[\lambda + n^2/\widetilde{\lambda}^2(z)]} \le \frac{W}{\min\limits_{\widetilde{y}\le z\le c}[\lambda + n^2/\widetilde{\lambda}^2(z)]} ,$$

and it can be made arbitrarily small by choosing a positive c close enough to 0. The second term in (3.16), for a fixed $c > 0$, converges to 0 as $\widetilde{y} \to 0+$. So we get that $\lim\limits_{\widetilde{y}\to 0+} \widetilde{g}(\widetilde{y}) = 0$.

Now we are going to find $D_{\widetilde{u}(\widetilde{y})}\widetilde{g}(0+)$. We have:

$$D_{\widetilde{u}(\widetilde{y})}\widetilde{g}(\widetilde{y}) = \frac{1}{W}\left[D_{\widetilde{u}(\widetilde{y})}\xi_1(\widetilde{y}) \int_{\widetilde{y}}^{b} \xi_2(z) \cdot G(z) d\widetilde{v}(z) + D_{\widetilde{u}(\widetilde{y})}\xi_2(\widetilde{y}) \int_0^{\widetilde{y}} \xi_1(z) \cdot G(z) d\widetilde{v}(z)\right] . \tag{3.17}$$

The first integral here is equal to $\displaystyle\int_{\widetilde{y}}^{c} + \int_c^b$, and it is not greater than

$$\|G\| \cdot [\xi_2(\widetilde{y}) \cdot \widetilde{v}(c) + \xi_2(c) \cdot \widetilde{v}(b)] ,$$

and the first summand is not greater than

$$\|G\|/W \cdot [W \cdot \widetilde{v}(c) + \xi_2(c) \cdot \widetilde{v}(b) \cdot D_{\widetilde{u}(\widetilde{y})}\xi_1(\widetilde{y})] .$$

By choosing $c \in (0, b)$ close enough to 0 we make $\widetilde{v}(c)$ arbitrarily small; and we know $D_{\widetilde{u}(\widetilde{y})}\xi_1(\widetilde{y}) \to 0$ as $\widetilde{y} \to 0+$. So the first summand in (3.17) goes to 0 as $\widetilde{y} \to 0+$.

The second summand in (3.17) does not exceed in absolute value

$$\|G\| \cdot \xi_1(\widetilde{y}) \cdot |D_{\widetilde{u}(\widetilde{y})}\xi_2(\widetilde{y})| \cdot \widetilde{v}(\widetilde{y}) \le \|G\| \cdot W \cdot \widetilde{v}(\widetilde{y}) \to 0 \quad (\widetilde{y} \to 0+) .$$

Now we are looking for the solution $g(\widetilde{y})$ of the equation (3.12) with the boundary conditions under this formula in the form $g(\widetilde{y}) = \widetilde{g}(\widetilde{y}) + C \cdot \xi_1(\widetilde{y})$. For the undetermined coefficient C we get one linear equation, and it does have a solution since $\xi_1(b) \neq 0$.

The same way we get, for $n \neq 0$, a solution $g(\widetilde{y})$ for $\widetilde{y} < 0$ with $g(0-) = D_{\widetilde{u}(\widetilde{y})} g(0-) = 0$, $g(a) = \mu^{-1} G(a)$.

So we get a solution $f \in D(A)$ of the equation (3.11) for every function $F(\theta, \widetilde{y}) = \sum_{n=-N}^{N} e^{in\theta} \cdot G_n(\widetilde{y})$, $G_n(\widetilde{y}) \in \mathbf{C}[a,b]$, such that $G_n(0) = 0$ for $n \neq 0$ (we take $f(\mathfrak{o}) = G_0(0)$). The set of such functions is dense in $\mathbf{C}(\mathfrak{C})$ so that the closure operator $\overline{(\lambda I - A|_{D(A)})^{-1}}$ is defined on the whole $\mathbf{C}(\mathfrak{C})$ which finishes the proof. $\square$

Let $\widetilde{\boldsymbol{q}}_t$ be the Markov process corresponding to $\overline{A|_{D(A)}}$, whose existence was proved in Lemma 3.1. We prove the following

Theorem 3.1. *As $\varepsilon \downarrow 0$, for fixed $T > 0$, the process $\widetilde{\boldsymbol{q}}_t^{\varepsilon} = \boldsymbol{\pi}(\boldsymbol{q}_t^{\varepsilon})$ converges weakly in the space $\mathbf{C}_{[0,T]}(\mathfrak{C})$ to the process $\widetilde{\boldsymbol{q}}_t$.*

The proof is again based on an application of Lemma 2.2.

Proof of Theorem 3.1. Making use of Lemma 2.2, we take the metric space $M = S^1 \times [a-1, b+1]$ with standard metric. The mapping $Y = \boldsymbol{\pi}$. The space $Y(M) = \mathfrak{C}$ is endowed with the metric d, defined as follows. For any two points $(\theta_1, \widetilde{y}_1)$ and $(\theta_2, \widetilde{y}_2)$ on $\mathfrak{C}$ with $\widetilde{y}_1, \widetilde{y}_2$ having the same sign we let $d((\theta_1, \widetilde{y}_1), (\theta_2, \widetilde{y}_2))$ be the Euclidean distance between points $(|\widetilde{y}_1| \cos\theta_1, |\widetilde{y}_1| \sin\theta_1)$ and $(|\widetilde{y}_2| \cos\theta_2, |\widetilde{y}_2| \sin\theta_2)$ in $\mathbb{R}^2$; if $\widetilde{y}_1$ and $\widetilde{y}_2$ have different sign we take $d((\theta_1, \widetilde{y}_1), (\theta_2, \widetilde{y}_2)) = d((\theta_1, \widetilde{y}_1), \mathfrak{o}) + d(\mathfrak{o}, (\theta_2, \widetilde{y}_2))$. With respect to this metric the space $\mathfrak{C}$ is a complete separable metric space. We take the process $(X_t^{\varepsilon}, \mathbf{P}_x^{\varepsilon})$ as $\boldsymbol{q}_t^{\varepsilon}$ and the process $(y_t, \mathbf{P}_y)$ is taken as $\widetilde{\boldsymbol{q}}_t$.

For the uniqueness of solution of martingale problem we set the space Ψ be the space of all continuous functions on $\mathfrak{C}$ which has the form $F(\theta, \widetilde{y}) = \sum_{n=-N}^{N} e^{in\theta} \cdot G_n(\widetilde{y})$, $G_n \in \mathbf{C}[a,b]$ is continuously differentiable inside $[a, 0)$ and $(0, b]$, also $G_n(0) = 0$ for $n \neq 0$. We take $f(\mathfrak{o}) = G_0(0)$. It is proved in the proof of Lemma 3.1 that the equation $\lambda f - Af = F$ always has a solution $f \in D \subset D(A)$ for all $F \in \Psi$ and $\lambda > 0$. The space D contains those functions $f \in \mathbf{C}(\mathfrak{C})$ that are bounded and are three times continuously differentiable inside $\mathfrak{C}^+ \equiv \{(\theta, \widetilde{y}) \in \mathfrak{C} : a < \widetilde{y} < 0\}$ and $\mathfrak{C}^- \equiv \{(\theta, \widetilde{y}) \in \mathfrak{C} : 0 < \widetilde{y} < b\}$.

We will state pre-compactness of family of distributions of processes $\widetilde{\boldsymbol{q}}_t^{\varepsilon}$ in Lemma 3.2. What remains to do is to check that for every compact $K \subset \mathfrak{C}$ and for every $f \in D$ and every $\lambda > 0$ we have

$$\mathbf{E}_{\boldsymbol{q}_0} \left[\int_0^{\infty} e^{-\lambda t} [\lambda f(\boldsymbol{\pi}(\boldsymbol{q}_t^{\varepsilon})) - Af(\boldsymbol{\pi}(\boldsymbol{q}_t^{\varepsilon}))] dt - f(\boldsymbol{\pi}(\boldsymbol{q}_0)) \right] \to 0$$

as $\varepsilon \downarrow 0$ uniformly in $\boldsymbol{q}_0 \in \boldsymbol{\pi}^{-1}(K)$. The proof of this is essentially the same as the proof we did in Lemma 2.5, based on the following auxiliary Lemmas 3.9 (for the proof

of convergence for processes near $\mathfrak{o}$) and 3.10 (for the proof of convergence for processes away from $\mathfrak{o}$) and the auxiliary Lemmas 2.9 and 2.10 (for the estimates on the exit times, notice that the stopping times σ_n and τ_n we will work with in this section are essentially the same stopping times that we worked with in Section 2 since we are discussing about a model problem). We omit the details in the proof. $\square$

Let κ be a real number with small absolute value. Let $G(\kappa) = \{(\theta, y) \in S^1 \times [a-1, b+1] : a-1 \leq y \leq -1-\kappa \text{ or } 1+\kappa \leq y \leq b+1\}$. Let $C^+(\kappa) = \{(\theta, y) \in S^1 \times [a-1, b+1] : y = 1+\kappa\}$ and $C^-(\kappa) = \{(\theta, y) \in S^1 \times [a-1, b+1] : y = -1-\kappa\}$. Let $C(\kappa) = C^+(\kappa) \cup C^-(\kappa)$. Let $\delta > \delta' > 0$ be small. We shall introduce a sequence of stopping times $\tau_0 \leq \sigma_0 < \tau_1 < \sigma_1 < \tau_2 < \sigma_2 < ...$ by

$$\tau_0 = 0 \ , \ \sigma_n = \min\{t \geq \tau_n, \boldsymbol{q}_t^\varepsilon \in G(\delta)\} \ , \ \tau_n = \min\{t \geq \sigma_{n-1}, \boldsymbol{q}_t^\varepsilon \in C(\delta')\} \ .$$

This is well-defined up to some σ_k ($k \geq 0$) such that

$$\mathbf{P}_{y^\varepsilon_{\sigma_k}}(y^\varepsilon_{t+\sigma_k} \text{ hits } a-1 \text{ or } b+1 \text{ before it hits } -1-\delta' \text{ or } 1+\delta') = 1 \ .$$

We will then define $\tau_{k+1} = \min\{t > \sigma_k : y_t^\varepsilon = a-1 \text{ or } b+1\}$. And we define $\tau_{k+1} < \sigma_{k+1} = \tau_{k+1} + 1 < \tau_{k+2} = \tau_{k+1} + 2 < \sigma_{k+2} = \tau_{k+1} + 3 < ...$ and so on.

We have $\lim\limits_{n\to\infty} \tau_n = \lim\limits_{n\to\infty} \sigma_n = \infty$. And we have obvious relations $\boldsymbol{q}^\varepsilon_{\tau_n} \in C(\delta')$, $\boldsymbol{q}^\varepsilon_{\sigma_n} \in C(\delta)$ for $1 \leq n \leq k$ (as long as $k \geq 1$, if $k = 0$ the process may start from $G(\delta)$ and goes directly to $S^1 \times \{a-1\}$ or $S^1 \times \{b+1\}$ without touching $C(\delta')$ and is stopped there, or it may start from $S^1 \times (-1-\delta, 1+\delta)$, reaches $C(\delta)$ first and then goes directly to $S^1 \times \{a-1\}$ or $S^1 \times \{b+1\}$ without touching $C(\delta')$ and is stopped there). Also, for $n \geq k+1$ we have $\boldsymbol{q}^\varepsilon_{\tau_n} = \boldsymbol{q}^\varepsilon_{\sigma_n} \in S^1 \times \{a-1\}$ or $S^1 \times \{b+1\}$. If $\boldsymbol{q}_0^\varepsilon = \boldsymbol{q}_0 \in G(\delta)$, then we have $\sigma_0 = 0$ and τ_1 is the first time at which the process $\boldsymbol{q}_t^\varepsilon$ reaches $C(\delta')$ or $S^1 \times \{a-1\}$ or $S^1 \times \{b+1\}$.

Note that these stopping times are the same as those defined in Section 2 since our process y_t^ε is essentially the process q_t^ε in Section 2.

The pre-compactness of the family $\{\widetilde{\boldsymbol{q}}_t^\varepsilon\}_{\varepsilon>0}$ in $\mathbf{C}_{[0,T]}(\mathfrak{C})$ for $0 < T < \infty$ is proved in the same way as in the one-dimensional case. We shall make use of the technical Lemma 2.3 with $\widetilde{q}_\bullet^{\varepsilon,\delta}$ and $\widetilde{q}_\bullet^\varepsilon$ replaced by $\widetilde{\boldsymbol{q}}_\bullet^{\varepsilon,\delta}$ and $\widetilde{\boldsymbol{q}}_\bullet^\varepsilon$ and the space $\mathbf{C}_{[0,T]}(\mathfrak{C})$ instead of $\mathbf{C}_{[0,T]}([a,b])$. We omit the proof of the next lemma.

Lemma 3.2. *The family of distributions of* $\{\widetilde{\boldsymbol{q}}_t^\varepsilon\}_{\varepsilon>0}$ *is pre-compact in* $\mathbf{C}_{[0,T]}(\mathfrak{C})$.

The next few lemmas establish the estimates on the asymptotic joint law of the processes $(y_t^\varepsilon, \theta_t^\varepsilon)$ at first exit from a small neighborhood of the domain within which the friction vanishes. This is the key part to the proof of Theorem 3.1.

Let $\delta'' > 0$ be small. We consider the process $\boldsymbol{q}_t^\varepsilon$ starting from $\boldsymbol{q}_0^\varepsilon = \boldsymbol{q}_0 \in S^1 \times [-1-\delta', 1+\delta']$. Let us introduce another sequence of stopping times $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < ... < \alpha_{n(\varepsilon)}$ by

$$\alpha_1 = \min\{0 \le t < \sigma_0 : \boldsymbol{q}_t^\varepsilon \in C(0)\} \ , \ \beta_1 = \min\{\alpha_1 < t < \sigma_0 : \boldsymbol{q}_t^\varepsilon \in C(-\delta'')\} \ ,$$

and for $k \ge 2$ we define

$$\alpha_k = \min\{\beta_{k-1} < t < \sigma_0 : \boldsymbol{q}_t^\varepsilon \in C(0)\} \ , \ \beta_k = \min\{\alpha_k < t < \sigma_0 : \boldsymbol{q}_t^\varepsilon \in C(-\delta'')\} \ .$$

Here we take the convention that the minimum over an empty set is ∞. The number $n(\varepsilon)$ is a non-negative integer-valued random variable such that $\alpha_{n(\varepsilon)} < \infty$ and $\beta_{n(\varepsilon)} = \infty$. If $\alpha_1 = \infty$ we set $n(\varepsilon) = 0$.

Lemma 3.3. *For $\boldsymbol{q}_0 \in G(\delta')$ we have*

$$\mathbf{P}_{\boldsymbol{q}_0}(\alpha_1 < \infty) \ge 1 - \max\left(\frac{\widetilde{u}(\delta') + \varepsilon\delta'}{\widetilde{u}(\delta) + \varepsilon\delta}, \frac{-\widetilde{u}(-\delta') + \varepsilon\delta'}{-\widetilde{u}(-\delta) + \varepsilon\delta}\right) \ . \tag{3.18}$$

Proof. If $1 \le y_0^\varepsilon = y_0 \le 1 + \delta'$ we have

$$\mathbf{P}_{\boldsymbol{q}_0}(\alpha_1 < \infty) = \frac{u^\varepsilon(1+\delta) - u^\varepsilon(y)}{u^\varepsilon(1+\delta) - u^\varepsilon(1)} \ge \frac{u^\varepsilon(1+\delta) - u^\varepsilon(1+\delta')}{u^\varepsilon(1+\delta) - u^\varepsilon(1)} = 1 - \frac{\widetilde{u}(\delta') + \varepsilon\delta'}{\widetilde{u}(\delta) + \varepsilon\delta} \ .$$

If $-1-\delta' \le y_0^\varepsilon = y_0 \le -1$ we have

$$\mathbf{P}_{\boldsymbol{q}_0}(\alpha_1 < \infty) = \frac{u^\varepsilon(y) - u^\varepsilon(-1-\delta')}{u^\varepsilon(-1) - u^\varepsilon(-1-\delta)} \ge \frac{u^\varepsilon(-1-\delta') - u^\varepsilon(-1-\delta)}{u^\varepsilon(-1) - u^\varepsilon(-1-\delta)} = 1 - \frac{-\widetilde{u}(-\delta') + \varepsilon\delta'}{-\widetilde{u}(-\delta) + \varepsilon\delta} \ .$$

If $-1 < y_0^\varepsilon = y_0 < 1$ we have $\mathbf{P}_{\boldsymbol{q}_0}(\alpha_1 < \infty) = 1$. $\square$

Lemma 3.4. *For $\boldsymbol{q}_0 \in G(\delta')$ we have*

$$\mathbf{P}_{\boldsymbol{q}_0}(\beta_1 < \infty | \alpha_1 < \infty) \ge 1 - \max\left(\frac{\varepsilon\delta''}{\widetilde{u}(\delta) + \varepsilon(\delta+\delta'')}, \frac{\varepsilon\delta''}{-\widetilde{u}(-\delta) + \varepsilon(\delta+\delta'')}\right) \ . \tag{3.19}$$

Proof. If $y_{\alpha_1}^\varepsilon = 1$ we have

$$\mathbf{P}_{\boldsymbol{q}_0}(\beta_1 < \infty | \alpha_1 < \infty) = \frac{u^{\varepsilon}(1+\delta) - u^{\varepsilon}(1)}{u^{\varepsilon}(1+\delta) - u^{\varepsilon}(1-\delta'')} = 1 - \frac{\varepsilon\delta''}{\widetilde{u}(\delta) + \varepsilon(\delta+\delta'')} \ .$$

If $y^{\varepsilon}_{\alpha_1} = -1$ we have

$$\mathbf{P}_{\boldsymbol{q}_0}(\beta_1 < \infty | \alpha_1 < \infty) = \frac{u^{\varepsilon}(-1) - u^{\varepsilon}(-1-\delta)}{u^{\varepsilon}(-1+\delta'') - u^{\varepsilon}(-1-\delta)} = 1 - \frac{\varepsilon\delta''}{-\widetilde{u}(-\delta) + \varepsilon(\delta+\delta'')} \ .$$

□

Let $M(\varepsilon) \to \infty$ as $\varepsilon \downarrow 0$ be an integer. The exact asymptotics of $M(\varepsilon)$ will be specified later. We prove

Lemma 3.5. *For $\boldsymbol{q}_0 \in G(\delta')$ we have*

$$\mathbf{P}_{\boldsymbol{q}_0}(n(\varepsilon) \geq M(\varepsilon) | \alpha_1 < \infty) \geq \left[1 - \max\left(\frac{\varepsilon\delta''}{\widetilde{u}(\delta) + \varepsilon(\delta+\delta'')}, \frac{\varepsilon\delta''}{-\widetilde{u}(-\delta) + \varepsilon(\delta+\delta'')}\right)\right]^{M(\varepsilon)-1} . \tag{3.20}$$

Proof. This is because trajectories of $\boldsymbol{q}_t^{\varepsilon}$ between times $\alpha_i \leq t < \alpha_{i+1}$ are independent and by iteratively using Lemma 3.4 we get the desired result. □

Lemma 3.6. *We have*

$$\alpha_{i+1} - \beta_i \geq \varepsilon^2 \left(\frac{\delta''}{H_i}\right)^5 \tag{3.21}$$

with H_i being i.i.d. positive random variables with $\mathbf{E}(H_i)^4 < \infty$ for $i = 1, 2, ..., n(\varepsilon) - 1$.

Proof. This is a result of the Hölder continuity of the standard Wiener trajectory $|W_t - W_s| \leq H_i |t-s|^{1/5}$ and the fact that between times $\beta_i \leq t < \alpha_{i+1}$ the process y_t^{ε} is a time-changed Wiener process $\dfrac{1}{\varepsilon} W_t$ traveling at least a distance of δ''. □

Let us define an auxiliary function

$$\begin{aligned} &\Omega(\varepsilon, \delta, \delta', \delta'', M(\varepsilon)) \\ &\equiv 2\left[1 - \left[1 - \max\left(\frac{\varepsilon\delta''}{\widetilde{u}(\delta) + \varepsilon(\delta+\delta'')}, \frac{\varepsilon\delta''}{-\widetilde{u}(-\delta) + \varepsilon(\delta+\delta'')}\right)\right]^{M(\varepsilon)-1} + \right. \\ &\qquad \left. 2\max\left(\frac{\widetilde{u}(\delta') + \varepsilon\delta'}{\widetilde{u}(\delta) + \varepsilon\delta}, \frac{-\widetilde{u}(-\delta') + \varepsilon\delta'}{-\widetilde{u}(-\delta) + \varepsilon\delta}\right)\right] \ . \end{aligned}$$

Lemma 3.7. *For $\boldsymbol{q}_0 \in G(\delta')$ and for some $A > 0$, $\kappa > 0$ and $C > 0$, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, for any $0 \le \theta_1 \le \theta_2 \le 2\pi$ we have*

$$\left|\mathbf{P}_{\boldsymbol{q}_0}(\theta^\varepsilon_{\sigma_0} \in [\theta_1, \theta_2], y^\varepsilon_{\sigma_0} = 1 + \delta) - \frac{\theta_2 - \theta_1}{2\pi}\mathbf{P}_{\boldsymbol{q}_0}(y^\varepsilon_{\sigma_0} = 1 + \delta)\right|$$
$$\le C \exp(-A(\delta'')^5 \kappa M(\varepsilon)) + 2\Omega(\varepsilon, \delta, \delta', \delta'', M(\varepsilon))$$

and

$$\left|\mathbf{P}_{\boldsymbol{q}_0}(\theta^\varepsilon_{\sigma_0} \in [\theta_1, \theta_2], y^\varepsilon_{\sigma_0} = -1 - \delta) - \frac{\theta_2 - \theta_1}{2\pi}\mathbf{P}_{\boldsymbol{q}_0}(y^\varepsilon_{\sigma_0} = -1 - \delta)\right|$$
$$\le C \exp(-A(\delta'')^5 \kappa M(\varepsilon)) + 2\Omega(\varepsilon, \delta, \delta', \delta'', M(\varepsilon)) \ .$$

Proof. As we have

$$x^\varepsilon_t = \int_0^t \frac{1}{\lambda(y^\varepsilon_t) + \varepsilon} dW^1_s = W^1\left(\int_0^t \frac{ds}{(\lambda(y^\varepsilon_s) + \varepsilon)^2}\right) ,$$

we set $T^\varepsilon(t) = \displaystyle\int_0^t \frac{ds}{(\lambda(y^\varepsilon_s) + \varepsilon)^2}$. Using Lemma 3.6 for $\boldsymbol{q}_0 \in G(\delta')$ the random time $T^\varepsilon(\sigma_0)$ can be estimated from below by

$$T^\varepsilon(\sigma_0) \ge \int_0^{\sigma_0} \frac{ds}{(\lambda(y^\varepsilon_s) + \varepsilon)^2} \ge \frac{1}{\varepsilon^2}\int_0^{\sigma_0} \mathbf{1}_{\{-1 \le y^\varepsilon_s \le 1\}} ds \ge \frac{1}{\varepsilon^2}\sum_{i=1}^{n(\varepsilon)-1} (\alpha_{i+1} - \beta_i) \ge (\delta'')^5 \sum_{i=1}^{n(\varepsilon)-1} \frac{1}{(H_i)^5} \ .$$

(If $n(\varepsilon) = 0, 1$ the sum is supposed to be 0.)

And we also notice that the random time $T^\varepsilon(\sigma_0)$ only depends on the behavior of the process y^ε_t and is therefore independent of the Wiener process W^1_t in the stochastic differential equation $\dot{x}^\varepsilon_t = \dfrac{1}{\lambda(y^\varepsilon_t) + \varepsilon}\dot{W}^1_t$ (see (3.2)). For the same reason the random variables $y^\varepsilon_{\sigma_0}$, $n(\varepsilon)$ and α_1 are of course also independent of W^1_t.

As we have the elementary inequality $\left(\mathbf{E}\dfrac{1}{(H_i)^5}\right)^{1/5} (\mathbf{E}(H_i)^4)^{1/4} \ge \left(\mathbf{E}\dfrac{1}{H_i}\right)(\mathbf{E}H_i) \ge 1$, we have, by Strong Law of Large Numbers

$$\lim_{\varepsilon \downarrow 0} \frac{1}{M(\varepsilon) - 1} \sum_{i=1}^{M(\varepsilon)-1} \frac{1}{(H_i)^5} = \mathbf{E}\left(\frac{1}{(H_i)^5}\right) \ge \frac{1}{(\mathbf{E}(H_i)^4)^{5/4}} \ge c > 0 \ \text{ a. s.}$$

for some constant $c > 0$. (We can always assume that H_i is uniformly bounded from below by a positive constant so that $\left(\mathbf{E}\dfrac{1}{(H_i)^5}\right) < \infty$ and we can apply SLLN.)

Now we see that we can find some $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ we will have

$$\mathbf{P}_{\boldsymbol{q}_0}(T^\varepsilon(\sigma_0) \ge (\delta'')^5 \kappa M(\varepsilon) | n(\varepsilon) \ge M(\varepsilon), \alpha_1 < \infty) = 1$$

for some constant $\kappa > 0$.

This gives

$$\begin{aligned}&\mathbf{P}_{\boldsymbol{q}_0}(T^\varepsilon(\sigma_0)\geq(\delta'')^5\kappa M(\varepsilon),y^\varepsilon_{\sigma_0}=1+\delta|n(\varepsilon)\geq M(\varepsilon),\alpha_1<\infty)\\&=\mathbf{P}_{\boldsymbol{q}_0}(y^\varepsilon_{\sigma_0}=1+\delta|n(\varepsilon)\geq M(\varepsilon),\alpha_1<\infty)\ .\end{aligned}$$

Recall that we have $\theta^\varepsilon_{\sigma_0}=x^\varepsilon_{\sigma_0}\mod 2\pi=W^1_{T^\varepsilon(\sigma_0)}\mod 2\pi$. Using this, the independence of $T^\varepsilon(\sigma_0)$, $y^\varepsilon_{\sigma_0}$, α_1 and $n(\varepsilon)$ with W^1_t, and the above estimates we have, as $0<\varepsilon<\varepsilon_0$, that

$$\begin{aligned}&\mathbf{P}_{\boldsymbol{q}_0}(\theta^\varepsilon_{\sigma_0}\in[\theta_1,\theta_2],y^\varepsilon_{\sigma_0}=1+\delta|n(\varepsilon)\geq M(\varepsilon),\alpha_1<\infty)\\&=\int_0^\infty\mathbf{P}_{\boldsymbol{q}_0}(T^\varepsilon(\sigma_0)\in dt,y^\varepsilon_{\sigma_0}=1+\delta|n(\varepsilon)\geq M(\varepsilon),\alpha_1<\infty)\mathbf{P}_{\boldsymbol{q}_0}(W^1_t\mod 2\pi\in[\theta_1,\theta_2])\\&=\int_{(\delta'')^5\lambda M(\varepsilon)}^\infty\mathbf{P}_{\boldsymbol{q}_0}(T^\varepsilon(\sigma_0)\in dt,y^\varepsilon_{\sigma_0}=1+\delta|n(\varepsilon)\geq M(\varepsilon),\alpha_1<\infty)\mathbf{P}_{\boldsymbol{q}_0}(W^1_t\mod 2\pi\in[\theta_1,\theta_2])\ .\end{aligned}$$

Since we have the exponential decay

$$\left|\mathbf{P}(W^1_t\mod 2\pi\in[\theta_1,\theta_2])-\frac{\theta_2-\theta_1}{2\pi}\right|<C\exp(-At)$$

for some $C>0$ and $A>0$, we could estimate

$$\begin{aligned}&\big|\mathbf{P}_{\boldsymbol{q}_0}(\theta^\varepsilon_{\sigma_0}\in[\theta_1,\theta_2]\ ,\ y^\varepsilon_{\sigma_0}=1+\delta|n(\varepsilon)\geq M(\varepsilon),\alpha_1<\infty)-\\&\qquad\frac{\theta_2-\theta_1}{2\pi}\,\mathbf{P}_{\boldsymbol{q}_0}(y^\varepsilon_{\sigma_0}=1+\delta|n(\varepsilon)\geq M(\varepsilon),\alpha_1<\infty)\big|\\&<C\exp(-A(\delta'')^5\kappa M(\varepsilon))\end{aligned}$$

for $0<\varepsilon<\varepsilon_0$.

Notice that we have, by using Lemmas 3.5 and 3.3,

$$\begin{aligned}&\big|\mathbf{P}_{\boldsymbol{q}_0}(\theta^\varepsilon_{\sigma_0}\in[\theta_1,\theta_2],y^\varepsilon_{\sigma_0}=1+\delta)-\mathbf{P}_{\boldsymbol{q}_0}(\theta^\varepsilon_{\sigma_0}\in[\theta_1,\theta_2]\ ,\ y^\varepsilon_{\sigma_0}=1+\delta|n(\varepsilon)\geq M(\varepsilon),\alpha_1<\infty)\big|\\&=\big|\mathbf{P}_{\boldsymbol{q}_0}(\theta^\varepsilon_{\sigma_0}\in[\theta_1,\theta_2],y^\varepsilon_{\sigma_0}=1+\delta|n(\varepsilon)\geq M(\varepsilon),\alpha_1<\infty)\mathbf{P}(n(\varepsilon)\geq M(\varepsilon),\alpha_1<\infty)-\\&\qquad\mathbf{P}_{\boldsymbol{q}_0}(\theta^\varepsilon_{\sigma_0}\in[\theta_1,\theta_2]\ ,\ y^\varepsilon_{\sigma_0}=1+\delta|n(\varepsilon)\geq M(\varepsilon),\alpha_1<\infty)\big|+\mathbf{P}_{\boldsymbol{q}_0}(n(\varepsilon)<M(\varepsilon))+\mathbf{P}_{\boldsymbol{q}_0}(\alpha_1=\infty)\\&\leq 2(\mathbf{P}_{\boldsymbol{q}_0}(n(\varepsilon)<M(\varepsilon))+\mathbf{P}_{\boldsymbol{q}_0}(\alpha_1=\infty))\\&\leq 2(\mathbf{P}_{\boldsymbol{q}_0}(n(\varepsilon)<M(\varepsilon)|\alpha_1<\infty)+2\mathbf{P}_{\boldsymbol{q}_0}(\alpha_1=\infty))\\&\leq 2\left[1-\left[1-\max\left(\frac{\varepsilon\delta''}{\widetilde{u}(\delta)+\varepsilon(\delta+\delta'')},\frac{\varepsilon\delta''}{-\widetilde{u}(-\delta)+\varepsilon(\delta+\delta'')}\right)\right]^{M(\varepsilon)-1}+\right.\\&\qquad\left.2\max\left(\frac{\widetilde{u}(\delta')+\varepsilon\delta'}{\widetilde{u}(\delta)+\varepsilon\delta},\frac{-\widetilde{u}(-\delta')+\varepsilon\delta'}{-\widetilde{u}(-\delta)+\varepsilon\delta}\right)\right]\\&=\Omega(\varepsilon,\delta,\delta',\delta'',M)\ .\end{aligned}$$

By the same argument we can estimate

$$\left|\frac{\theta_2-\theta_1}{2\pi}\mathbf{P}_{\boldsymbol{q}_0}(y^\varepsilon_{\sigma_0}=1+\delta)-\frac{\theta_2-\theta_1}{2\pi}\mathbf{P}_{\boldsymbol{q}_0}(y^\varepsilon_{\sigma_0}=1+\delta|n(\varepsilon)\geq M(\varepsilon),\alpha_1<\infty)\right|\leq\Omega(\varepsilon,\delta,\delta',\delta'',M)\ .$$

Summing up these estimates we have

$$
\begin{aligned}
&\left|\mathbf{P}_{\boldsymbol{q}_0}(\theta^\varepsilon_{\sigma_0}\in[\theta_1,\theta_2], y^\varepsilon_{\sigma_0}=1+\delta)-\frac{\theta_2-\theta_1}{2\pi}\mathbf{P}_{\boldsymbol{q}_0}(y^\varepsilon_{\sigma_0}=1+\delta)\right|\\
&\le \left|\mathbf{P}_{\boldsymbol{q}_0}(\theta^\varepsilon_{\sigma_0}\in[\theta_1,\theta_2], y^\varepsilon_{\sigma_0}=1+\delta)-\mathbf{P}_{\boldsymbol{q}_0}(\theta^\varepsilon_{\sigma_0}\in[\theta_1,\theta_2]\ ,\ y^\varepsilon_{\sigma_0}=1+\delta|n(\varepsilon)\ge M(\varepsilon),\alpha_1<\infty)\right|+\\
&\quad \Big|\mathbf{P}_{\boldsymbol{q}_0}(\theta^\varepsilon_{\sigma_0}\in[\theta_1,\theta_2]\ ,\ y^\varepsilon_{\sigma_0}=1+\delta|n(\varepsilon)\ge M(\varepsilon),\alpha_1<\infty)-\\
&\qquad\qquad \frac{\theta_2-\theta_1}{2\pi}\ \mathbf{P}_{\boldsymbol{q}_0}(y^\varepsilon_{\sigma_0}=1+\delta|n(\varepsilon)\ge M(\varepsilon),\alpha_1<\infty)\Big|+\\
&\quad \left|\frac{\theta_2-\theta_1}{2\pi}\mathbf{P}_{\boldsymbol{q}_0}(y^\varepsilon_{\sigma_0}=1+\delta)-\frac{\theta_2-\theta_1}{2\pi}\mathbf{P}_{\boldsymbol{q}_0}(y^\varepsilon_{\sigma_0}=1+\delta|n(\varepsilon)\ge M(\varepsilon),\alpha_1<\infty)\right|\\
&\le 2\Omega(\varepsilon,\delta,\delta',\delta'',M)+C\exp(-A(\delta'')^5\kappa M(\varepsilon))\ ,
\end{aligned}
$$

as desired. The other inequality is established in a similar way. $\square$

Combining Lemma 3.7 and Lemma 2.7 we can have

Lemma 3.8. *For $\boldsymbol{q}_0\in G(\delta')$ and for some $A>0$, $\kappa>0$ and $C_1,C_2>0$, there exists $\varepsilon_0>0$ such that for all $0<\varepsilon<\varepsilon_0$, for any $0\le\theta_1\le\theta_2\le 2\pi$ we have*

$$
\begin{aligned}
&\left|\mathbf{P}_{\boldsymbol{q}_0}(\theta^\varepsilon_{\sigma_0}\in[\theta_1,\theta_2], y^\varepsilon_{\sigma_0}=1+\delta)-\frac{\theta_2-\theta_1}{2\pi}\frac{\widetilde{u}(0)-\widetilde{u}(-\delta)}{\widetilde{u}(\delta)-\widetilde{u}(-\delta)}\right|\\
&\le C_1\exp(-A(\delta'')^5\kappa M(\varepsilon))+2\Omega(\varepsilon,\delta,\delta',\delta'',M(\varepsilon))+\frac{\widetilde{u}(\delta')-\widetilde{u}(0)+C_2\varepsilon}{\widetilde{u}(\delta)-\widetilde{u}(-\delta)}\equiv\rho(\varepsilon)\ ,
\end{aligned}
\tag{3.22}
$$

and

$$
\begin{aligned}
&\left|\mathbf{P}_{\boldsymbol{q}_0}(\theta^\varepsilon_{\sigma_0}\in[\theta_1,\theta_2], y^\varepsilon_{\sigma_0}=-1-\delta)-\frac{\theta_2-\theta_1}{2\pi}\frac{\widetilde{u}(\delta)-\widetilde{u}(0)}{\widetilde{u}(\delta)-\widetilde{u}(-\delta)}\right|\\
&\le C_1\exp(-A(\delta'')^5\kappa M(\varepsilon))+2\Omega(\varepsilon,\delta,\delta',\delta'',M(\varepsilon))+\frac{\widetilde{u}(\delta')-\widetilde{u}(0)+C_2\varepsilon}{\widetilde{u}(\delta)-\widetilde{u}(-\delta)}\equiv\rho(\varepsilon)\ .
\end{aligned}
\tag{3.23}
$$

Now let us specify the asymptotic order of $M(\varepsilon)\to\infty$, $\delta=\delta(\varepsilon)\to 0$, $\delta'=\delta'(\varepsilon)\to 0$ and $\delta''=\delta''(\varepsilon)\to 0$ as $\varepsilon\downarrow 0$. Since for $0<\kappa<1$ we have the elementary estimate $1-(1-\kappa)^n=\kappa(1+(1-\kappa)+...+(1-\kappa)^{n-1})\le\kappa n$ we can estimate

$$
\begin{aligned}
&\Omega(\varepsilon,\delta,\delta',\delta'',M(\varepsilon))\\
&\le 2\left[M(\varepsilon)\cdot\max\left(\frac{\varepsilon\delta''}{\widetilde{u}(\delta)+\varepsilon(\delta+\delta'')},\frac{\varepsilon\delta''}{-\widetilde{u}(-\delta)+\varepsilon(\delta+\delta'')}\right)+\right.\\
&\qquad\left. 2\max\left(\frac{\widetilde{u}(\delta')+\varepsilon\delta'}{\widetilde{u}(\delta)+\varepsilon\delta},\frac{-\widetilde{u}(-\delta')+\varepsilon\delta'}{-\widetilde{u}(-\delta)+\varepsilon\delta}\right)\right]\ .
\end{aligned}
$$

We shall choose $\delta''=\delta''(\varepsilon)<<\delta$ and $M(\varepsilon)$ such that the requirements of Lemmas 2.6, 2.7 and 2.8 hold. At the same time, we need

$$
(\delta'')^5M(\varepsilon)\gtrsim\ln\frac{1}{(\widetilde{u}(\delta)-\widetilde{u}(-\delta))^2}
\tag{3.24}
$$

and

$$M(\varepsilon)\frac{\varepsilon\delta''}{\widetilde{u}(\delta)\wedge(-\widetilde{u}(-\delta))}\lesssim(\widetilde{u}(\delta)-\widetilde{u}(-\delta))^2\ . \tag{3.25}$$

To this end we let $M(\varepsilon)=\ln\left(\frac{1}{\varepsilon}\right)$ and $\delta''=\left(\frac{(\frac{1}{\widetilde{u}(\delta)-\widetilde{u}(-\delta)})\ln(\frac{1}{\widetilde{u}(\delta)-\widetilde{u}(-\delta)})^2}{\ln(\frac{1}{\varepsilon})}\right)^{1/5}$. At the same time we keep our asymptotic order of choice of ε, δ and δ' as in Section 2. This means that we need

$$\varepsilon\left(\ln\left(\frac{1}{\varepsilon}\right)\right)^{4/5}\frac{1}{\widetilde{u}(\delta)-\widetilde{u}(-\delta)}\ln\left(\frac{1}{\widetilde{u}(\delta)-\widetilde{u}(-\delta)}\right)^2\lesssim(\widetilde{u}(\delta)-\widetilde{u}(-\delta))^2\ .$$

It could be checked that this is possible to make (3.24) and (3.25) to hold. We formulate this as a corollary.

Corollary 3.1. *Let $\boldsymbol{q}_0\in G(\delta')$. Under the above specified asymptotic order we have, there exist $\varepsilon_0>0$ such that for any $0<\varepsilon<\varepsilon_0$ we have*

$$\left|\mathbf{P}_{\boldsymbol{q}_0}(\theta^\varepsilon_{\sigma_0}\in[\theta_1,\theta_2],y^\varepsilon_{\sigma_0}=1+\delta)-\frac{\theta_2-\theta_1}{2\pi}\frac{\widetilde{u}(0)-\widetilde{u}(-\delta)}{\widetilde{u}(\delta)-\widetilde{u}(-\delta)}\right|\leq C\cdot(\widetilde{u}(\delta)-\widetilde{u}(-\delta))^2\ , \tag{3.26}$$

$$\left|\mathbf{P}_{\boldsymbol{q}_0}(\theta^\varepsilon_{\sigma_0}\in[\theta_1,\theta_2],y^\varepsilon_{\sigma_0}=-1-\delta)-\frac{\theta_2-\theta_1}{2\pi}\frac{\widetilde{u}(\delta)-\widetilde{u}(0)}{\widetilde{u}(\delta)-\widetilde{u}(-\delta)}\right|\leq C\cdot(\widetilde{u}(\delta)-\widetilde{u}(-\delta))^2\ . \tag{3.27}$$

Lemma 3.9. *For any $\boldsymbol{q}\in G(\delta')$ and for any $\rho>0$ there exist $\varepsilon_0=\varepsilon_0(\rho)$ such that for any $0<\varepsilon<\varepsilon_0$, for any $f\in D(A)$ we have, for some $K>0$*

$$|\mathbf{E}_{\boldsymbol{q}}f(\boldsymbol{\pi}(\boldsymbol{q}^\varepsilon_{\sigma_0}))-f(\boldsymbol{\pi}(\boldsymbol{q}))|<K(\widetilde{u}(\delta)-\widetilde{u}(-\delta))^2\ . \tag{3.28}$$

Proof. We have, using Corollary 3.1, that

$$\begin{aligned}
&|\mathbf{E}_{\boldsymbol{q}}f(\boldsymbol{\pi}(\boldsymbol{q}^\varepsilon_{\sigma_0}))-f(\boldsymbol{\pi}(\boldsymbol{q}))|\\
&=\left|\mathbf{E}_{\boldsymbol{q}}f(\theta^\varepsilon_{\sigma_0},\pi(y^\varepsilon_{\sigma_0}))-f(\boldsymbol{\pi}(\boldsymbol{q}))\right|\\
&=\left|\int_0^{2\pi}f(\theta,\delta)\mathbf{P}_{\boldsymbol{q}}(\theta^\varepsilon_{\sigma_0}\in d\theta,y^\varepsilon_{\sigma_0}=1+\delta)+\int_0^{2\pi}f(\theta,-\delta)\mathbf{P}_{\boldsymbol{q}}(\theta^\varepsilon_{\sigma_0}\in d\theta,y^\varepsilon_{\sigma_0}=-1-\delta)-f(\boldsymbol{\pi}(\boldsymbol{q}))\right|\\
&\leq\left|\frac{1}{2\pi}\int_0^{2\pi}\frac{\widetilde{u}(0)-\widetilde{u}(-\delta)}{\widetilde{u}(\delta)-\widetilde{u}(-\delta)}f(\theta,\delta)d\theta+\frac{1}{2\pi}\int_0^{2\pi}\frac{\widetilde{u}(\delta)-\widetilde{u}(0)}{\widetilde{u}(\delta)-\widetilde{u}(-\delta)}f(\theta,-\delta)d\theta-f(\boldsymbol{\pi}(\boldsymbol{q}))\right|+K_1(\widetilde{u}(\delta)-\widetilde{u}(-\delta))^2\\
&=\left|\frac{1}{2\pi}\int_0^{2\pi}\frac{\widetilde{u}(0)-\widetilde{u}(-\delta)}{\widetilde{u}(\delta)-\widetilde{u}(-\delta)}(f(\theta,\delta)-f(\mathfrak{o}))d\theta-\right.\\
&\qquad\left.\frac{1}{2\pi}\int_0^{2\pi}\frac{\widetilde{u}(\delta)-\widetilde{u}(0)}{\widetilde{u}(\delta)-\widetilde{u}(-\delta)}(f(\mathfrak{o})-f(\theta,-\delta))d\theta+(f(\mathfrak{o})-f(\boldsymbol{\pi}(\boldsymbol{q})))\right|+K_1(\widetilde{u}(\delta)-\widetilde{u}(-\delta))^2\\
&\leq\left|\frac{(\widetilde{u}(0)-\widetilde{u}(-\delta))(\widetilde{u}(\delta)-\widetilde{u}(0))}{\widetilde{u}(\delta)-\widetilde{u}(-\delta)}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{f(\theta,\delta)-f(\mathfrak{o})}{\widetilde{u}(\delta)-\widetilde{u}(0)}d\theta-\frac{1}{2\pi}\int_0^{2\pi}\frac{f(\mathfrak{o})-f(\theta,-\delta)}{\widetilde{u}(0)-\widetilde{u}(-\delta)}d\theta\right)\right|+\\
&\qquad|f(\mathfrak{o})-f(\boldsymbol{\pi}(\boldsymbol{q}))|+K_1(\widetilde{u}(\delta)-\widetilde{u}(-\delta))^2\\
&\leq K(\widetilde{u}(\delta)-\widetilde{u}(-\delta))^2
\end{aligned}$$

for some $K_1 > 0$ and $K > 0$. We have used the gluing condition (3.10) and our specified choice of asymptotic order of δ, δ' and ε. $\square$

Lemma 3.10. *We have, as $\varepsilon, \delta, \delta'$ are small, for $\boldsymbol{q}_0 \in G(\delta)$, that*

$$\left|\mathbf{E}_{\boldsymbol{q}_0}\left[\int_{\sigma_0}^{\tau_1} e^{-\lambda t}[\lambda f(\boldsymbol{\pi}(\boldsymbol{q}_t^\varepsilon)) - Af(\boldsymbol{\pi}(\boldsymbol{q}_t^\varepsilon))]dt + e^{-\lambda\tau_1} f(\boldsymbol{\pi}(\boldsymbol{q}_{\tau_1}^\varepsilon))\right] - f(\boldsymbol{\pi}(\boldsymbol{q}_0))\right| \leq (\widetilde{u}(\delta) - \widetilde{u}(-\delta))^2 \ . \tag{3.29}$$

The proof of this Lemma is essentially the same proof in Lemma 2.6 modified into a two-dimensional version and we omit it.

Finally we would like to mention that our boundary condition given in this section also appears naturally in other model problems. As an example let consider the following system:

$$\begin{cases} x_t^\varepsilon = \displaystyle\int_0^t \frac{1}{\lambda(y_t^\varepsilon) + \varepsilon} dW_t^1 \ , \\ y_t^\varepsilon = |W_t^2| \ . \end{cases} \tag{3.30}$$

Here $\lambda(\bullet)$ is a smooth function on $\mathbb{R}_+$ that vanishes at 0 and is strictly positive in $(0, \infty)$; W_t^1 and W_t^2 are two independent standard Wiener processes on $\mathbb{R}$. Let the process $z_t^\varepsilon = (x_t^\varepsilon, y_t^\varepsilon)$ on $\mathbb{R} \times \mathbb{R}_+$ be stopped once it hits the boundary $\{(x, y) \in \mathbb{R}^2 : y = R\}$ for some $R > 0$. Let $\theta_t^\varepsilon = x_t^\varepsilon \mod 2\pi$. Let $\boldsymbol{\pi} : S^1 \times \mathbb{R}_+ \to \mathbb{R}^2$ be the mapping defined by $\boldsymbol{\pi}(\theta, y) = (y\cos\theta, y\sin\theta)$. For each fixed $\varepsilon > 0$, the process $w_t^\varepsilon = (\theta_t^\varepsilon, y_t^\varepsilon)$ is a diffusion process on $S^1 \times [0, R]$ with normal reflection at the boundary $\{(\theta, y) : y = 0\}$ and is stopped once it hits the other boundary $\{(\theta, y) : y = R\}$. Let $m_t^\varepsilon = \boldsymbol{\pi}(w_t^\varepsilon)$ (i.e., we glue all points $\{(\theta, y) : y = 0\}$). The process m_t^ε moves within the disk $B(R) = \{m \in \mathbb{R}^2 : |m|_{\mathbb{R}^2} \leq R\}$ and is stopped once it hits the boundary. In general, this process is *not* a Markov process. But we expect that, as $\varepsilon \downarrow 0$, this process w_t^ε will converge weakly to a Markov process w_t on $B(R)$ with generator A and the domain of definition $D(A)$, defined as follows: The operator A at points (θ, r) (we use polar coordinates, that is, a point $(x, y) \in \mathbb{R}^2$ is represented by $(r\cos\theta, r\sin\theta)$) with $r \neq 0$ is defined by

$$Af(\theta, r) = \frac{1}{2\lambda^2(r)} \frac{\partial^2}{\partial\theta^2} f(\theta, r) + \frac{1}{2} \frac{\partial^2}{\partial r^2} f(\theta, r) \ . \tag{3.31}$$

The domain of definition $D(A)$ of the operator A consists of those continuous functions f on $B(R)$ for which $Af(\theta, r)$ is defined and continuous for $r \neq 0$, the derivative in r being continuous; such that finite limit

$$\lim_{\theta' \to \theta, r \to 0+} \frac{\partial f}{\partial r}(\theta', r) \tag{3.32}$$

exists;

$$\lim_{\theta'\to\theta, r\to 0+} Af(\theta', r) \tag{3.33}$$

exists and does not depend on θ;

$$\lim_{\theta'\to\theta, r\to R-} Af(\theta', r) = 0 \ ; \tag{3.34}$$

and

$$\int_0^{2\pi} \lim_{\theta'\to\theta, r\to 0+} \frac{\partial f}{\partial r}(\theta', r) d\theta = 0 \ . \tag{3.35}$$

We define, for $f \in D(A)$, $Af(\theta, R)$ as the limit (3.34) and $Af(O)$ as the limit (3.33).

The weak convergence of w_t^ε to w_t in $\mathbf{C}_{[0,T]}(B(R))$ described above shall be a result of fast motion x_t^ε running at the local time of the slow motion y_t^ε on the boundary $\{(x, y) \in \mathbb{R} \times \mathbb{R}_+ : y = 0\}$. The proof of this result shall follow the same method of this section.

4 A conjecture in the general multidimensional case

In this section we give a conjecture in the general multidimensional case. Consider the general multidimensional problem (1.8), and for brevity assume that $\boldsymbol{b}(\bullet) \equiv \mathbf{0}$. That is, the system has the form

$$\dot{\boldsymbol{q}}_t^\varepsilon = -\frac{\nabla\lambda(\boldsymbol{q}_t^\varepsilon)}{2(\lambda(\boldsymbol{q}_t^\varepsilon)+\varepsilon)^3} + \frac{1}{\lambda(\boldsymbol{q}_t^\varepsilon)+\varepsilon}\dot{\boldsymbol{W}}_t \ , \ \boldsymbol{q}_0^\varepsilon = \boldsymbol{q} \in \mathbb{R}^d \ . \tag{4.1}$$

Let us work in a large closed ball $B(R) = \{\boldsymbol{q} \in \mathbb{R}^d : |\boldsymbol{q}|_{\mathbb{R}^d} \leq R\}$ for some $R > 0$, i.e., the process $\boldsymbol{q}_t^\varepsilon$ is stopped once it hits $\partial B(R)$. Suppose the friction $\lambda(\bullet)$ is smooth and $\lambda(\boldsymbol{q}) = 0$ for $\boldsymbol{q}$ in some region $G \subset B(R)$ while $\lambda(\boldsymbol{q}) > 0$ for $\boldsymbol{q} \in B(R)\backslash[G]$ (here $[G]$ is the closure of G with respect to the Euclidean metric in $\mathbb{R}^d$). The domain $G \subset B(R)$ is assumed to be simply connected and to have a smooth boundary ∂G.

Let $\mathfrak{C}$ be a topological space consisting of all points in $B(R)\backslash[G]$ and one additional point $\mathfrak{o}$. The topology of $\mathfrak{C}$ contains all the open subsets (in standard Euclidean metric) in the induced topology of $B(R)\backslash[G]$ and all the open neighborhoods of $[G]$ in $B(R)$ as the open subsets of $\mathfrak{C}$ containing $\mathfrak{o}$. Let us consider a projection $\boldsymbol{\pi} : B(R) \to \mathfrak{C}$ defined as follows: for points $\boldsymbol{q} \in B(R)\backslash[G]$ we have $\boldsymbol{\pi}(\boldsymbol{q}) = \boldsymbol{q}$ and for points $\boldsymbol{q} \in [G]$ we have $\boldsymbol{\pi}(\boldsymbol{q}) = \mathfrak{o}$. Under the above defined topology for $\mathfrak{C}$ the mapping $\boldsymbol{\pi}$ is continuous. Let $\widetilde{\boldsymbol{q}}_t^\varepsilon = \boldsymbol{\pi}(\boldsymbol{q}_t^\varepsilon)$ be a stochastic process with continuous trajectories on $\mathfrak{C}$.

Our conjecture is about the weak convergence, as $\varepsilon \downarrow 0$, of $\widetilde{\boldsymbol{q}}_t^\varepsilon$ to some Markov process $\widetilde{\boldsymbol{q}}_t$ on $\mathfrak{C}$. Below we give our definition of the latter process but we point out that we are not clear about the *existence* of it. Our generator and boundary condition for this process is more or less in the spirit of martingale problems (see, for example, [3, Ch.4]).

To ensure the *uniqueness* of solution of martingale problems we need the *existence* of solution in a nice space of the corresponding PDE with the specified boundary condition. We are not clear about this yet.

The process $\widetilde{\boldsymbol{q}}_t$ is identified by its generator A with domain of definition $D(A)$. For a function $f(\widetilde{\boldsymbol{q}})$ on $\mathfrak{C}$ that is continuous on $\mathfrak{C}$ and smooth for $\widetilde{\boldsymbol{q}} \neq \mathfrak{o}$, $|\widetilde{\boldsymbol{q}}|_{\mathbb{R}^d} < R$ we define

$$Af(\widetilde{\boldsymbol{q}}) = -\frac{\nabla\lambda(\widetilde{\boldsymbol{q}}) \cdot \nabla f(\widetilde{\boldsymbol{q}})}{2\lambda^3(\widetilde{\boldsymbol{q}})} + \frac{1}{2\lambda^2(\widetilde{\boldsymbol{q}})}\Delta f(\widetilde{\boldsymbol{q}}) \ , \tag{4.2}$$

and at the points $\mathfrak{o}$ and $\widetilde{\boldsymbol{q}}$ with $|\widetilde{\boldsymbol{q}}|_{\mathbb{R}^d}$ we define the values of Af as the limits of the values given by (4.2) (assuming these limits exist). The domain $D(A)$ is defined as the set of functions f such that $Af(\widetilde{\boldsymbol{q}}) = 0$ for $|\widetilde{\boldsymbol{q}}|_{\mathbb{R}^d} = R$, the generalized normal derivative

$$D_{\widetilde{u}}f(\boldsymbol{q}) = \lim_{\delta\downarrow 0} \frac{f(\boldsymbol{q} + \delta\boldsymbol{n}(\boldsymbol{q})) - f(\mathfrak{o})}{\widetilde{u}(\boldsymbol{q} + \delta\boldsymbol{n}(\boldsymbol{q}))} \tag{4.3}$$

exists for all $\boldsymbol{q} \in \partial G$, where $\boldsymbol{n}(\boldsymbol{q})$ is the vector of the outside normal to ∂G, and $\widetilde{u}(\boldsymbol{q})$ is some function defined in a neighborhood of ∂G with $\lim\limits_{\boldsymbol{\pi}(\boldsymbol{q})\to\mathfrak{o}} \widetilde{u}(\boldsymbol{q}) = 0$; and

$$\int_{\partial G} D_{\widetilde{u}}f(\boldsymbol{q})d\sigma(\boldsymbol{q}) = 0 \ . \tag{4.4}$$

Here $d\sigma(\boldsymbol{q})$ denotes integration with respect to the surface area on ∂G.

Conjecture. *The process $\widetilde{\boldsymbol{q}}_t^\varepsilon = \boldsymbol{\pi}(\boldsymbol{q}_t^\varepsilon)$ converges weakly in the space $\mathbf{C}_{[0,T]}(\mathfrak{C})$ as $\varepsilon \downarrow 0$ to the process $\widetilde{\boldsymbol{q}}_t$ described above.*

A further conjecture: we can define the function $\widetilde{u}$ as

$$\widetilde{u}(\boldsymbol{q} + \delta\boldsymbol{n}(\boldsymbol{q})) = \int_0^\delta \lambda(\boldsymbol{q} + s\boldsymbol{n}(\boldsymbol{q}))ds \tag{4.5}$$

for $\boldsymbol{q} \in \partial G$ and $\delta > 0$ sufficiently small.

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References

[1] Barlow, M., Pitman, J., Yor, M., On Walsh's Brownian motion, *Séminaire de Probabilitiés XXIII*, Springer Lecture Notes in Mathematics, **1372** (1989), pp. 275 – 293.

[2] Dynkin, E., One dimensional continuous strong Markov processes, *Theory of Probability and Its Applications* (English translation), **4**, 1, 1959, pp. 1 – 52.

[3] Ethier, S., Kurtz, T., *Markov processes, characterization and convergence*, John Wiley and Sons, London, 1986.

[4] Feller, W., Generalized second-order differential operators and their lateral conditions, *Ill. Journal of Math.*, **1** (1957), pp. 459 – 504.

[5] Freidlin, M., Some Remarks on the Smoluchowski-Kramers Approximation, *Journal of Statistical Physics*, **117**, No.314, pp. 617 – 634, 2004.

[6] Freidlin, M., and Hu, W., Smoluchowski-Kramers approximation in the case of variable friction, *Journal of Mathematical Sciences*, **79**, No.1, November 2011, pp. 184 – 207, translated from *Problems in Mathematical Analysis*, **61**, October 2011 (in Russian).

[7] Freidlin, M., and Wentzell, A., On the Neumann problem for PDE's with a small parameter and the corresponding diffusion processes, *Probability Theory and Related Fields*, online, DOI:10.1007/s00440-010-0317-4.

[8] Freidlin, M., and Wentzell, A., *Random Perturbations of Dynamical Systems*, 2-nd edition, Springer, 1998.

[9] Freidlin, M., and Wentzell, A., Random Perturbations of Hamiltonian Systems, *Mem. of AMS*, **109** (1994).

[10] Freidlin, M., and Wentzell, A., Diffusion processes on graphs and the averaging principle, *Annals of Probability*, **21**, 4, 1993, pp. 2215 – 2245.

[11] Mandl, P., *Analytical Treatment of One-dimensional Markov Processes*, Springer, 1968.

[12] Mochanov, S., On a problem in the theory of diffusion processes, *Theory of Probability and Its Applications* (English translation), **9**, 1964, pp. 472 – 477.

[13] Mochanov, S., and Ostrovskii, E., Symmetric stable processes as traces of degenerate diffusion processes, *Theory of Probability and Its Applications* (English translation), **14**, 1969, pp. 128 – 131.

[14] Revuz, D., and Yor, M., *Continuous Martingales and Brownian Motion*, 3.ed., Springer, 1999.

[15] Ueno, T., The diffusion satisfying Wentzell's boundary condition and the Markov process on the boundary I, *Proc. Japan Acad.*, **36**, 10, 1960, pp. 533 – 538.

[16] Ueno, T., The diffusion satisfying Wentzell's boundary condition and the Markov process on the boundary II, *Proc. Japan Acad.*, **36**, 10, 1960, pp. 625 – 629.

[17] Wentzell, A., On boundary conditions for multidimensional diffusion processes, *Theory of Probability and Its Applications* (English translation), **4**, 2, 1959, pp. 164 – 177.